VECTOR ANALYSIS: THE FORCE TABLE

OBJECT: To acquaint the students with the first condition of equilibrium and the analysis of vectors (forces) by graphical and analytical methods.

APPARATUS: Force table, ring, pulleys with attachments, string, hangers, weights, protractor and triple.

1. INTRODUCTION

Quantities that can be specified by giving their magnitudes are called SCALAR quantities (temperature, length, mass, volume, etc.). Quantities that require both a magnitude and a direction in their proper description are called VECTOR quantities. Examples of vector quantities are velocity, force, acceleration, etc. The mathematical treatment of scalar quantities is readily understood. For example, the sum of a volume of 3.0cm³ and a volume of 5.0cm³ is a volume of 8.0cm³. However, the mathematical treatment of vector quantities is more complicated and is the subject of this experiment. In brief, vectors may be added and subtracted; one vector can be multiplied by another vector in two different ways but the division of one vector by another is not generally defined. Here, we shall learn multiplying vectors \( \mathbf{A} \times \mathbf{B} \) called the vector or cross product, and \( \mathbf{A} \cdot \mathbf{B} \) called the scalar or dot product will be discussed in your lecture class.

The addition and subtraction of vectors in one dimension is straightforward i.e. if several vectors \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) etc., are acting in the same direction, their sum is \( \mathbf{A} + \mathbf{B} + \mathbf{C} \). However, if one of these vectors, say \( \mathbf{C} \) acts in a direction opposite to \( \mathbf{A} \) and \( \mathbf{B} \), then either \( \mathbf{C} \) must be subtracted from \( \mathbf{A} + \mathbf{B} \) i.e. \( \mathbf{A} + \mathbf{B} - \mathbf{C} \) or \( -\mathbf{C} \) must be added to \( \mathbf{A} + \mathbf{B} \) i.e. \( \mathbf{A} + \mathbf{B} + (-\mathbf{C}) \). The addition/subtraction of vector in two and three-dimensions requires the resolution of vectors along X-, Y- and Z- axes (analytic method) or all the vectors should be plotted on a graph paper to determine the result (graphical method). Both of these methods are used in this lab and are described below.

1.1 GRAPHICAL METHOD: A vector can be represented by a 'directed line' or an 'arrow'. The magnitude of the vector can be shown by choosing a convenient scale. For example, Fig. 1 shows two forces, \( \mathbf{F}_1 = 3.0 \text{N at } 45^\circ \) and \( \mathbf{F}_2 = 4.0 \text{N at } 120^\circ \), originating from the same point and the scale is 1.0 inch = 1.0N. The same procedure can be used to represent a system of several vectors.

1.1.1 Vector Addition: Fig. 2 illustrates the method of vector addition using vectors \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) from Fig. 1. Starting at point P, draw the first vector \( \mathbf{F}_1 \) in its own direction with its length proportional to its magnitude. Then draw the second vector \( \mathbf{F}_2 \) which begins at the ending of the first vector but in its own direction and with its length proportional to its magnitude. The sum of \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) is the new vector \( \mathbf{R} \) which is drawn from the beginning of the first vector (point P) to the ending of the second vector (point Q). The vector \( \mathbf{R} \) is called the RESULTANT with its magnitude proportional to the length \( PQ \) and direction shown by the arrow. If three or more vectors act at a single point (Fig. 3), then their vector sum can be found in the same manner (Fig. 4). It is important to note that the magnitude and direction of the resultant \( \mathbf{R} \) are independent of the order in which the original vectors are drawn. It should be realized that graphical procedures usually introduce errors arising from inaccuracies in drawing various arrows representing the vectors and that one can determine this error (known as CLOSURE ERROR).
1.2 ANALYTIC METHOD: In order to treat vectors analytically, one needs to be fully familiar with basic Trigonometry to resolve a vector into components. For a vector $\mathbf{F}$ directed at angle $\theta$ upward from the horizontal, the X and Y components are $F_x = F\cos\theta$ and $F_y = F\sin\theta$. The vector sum can be determined as follows. Let $F_1, F_2, F_3...$ represent the forces (vectors) to be added vectorially. First, each vector is resolved into X and Y components. Then the vector sum $\mathbf{R}$ can be written in its X- and Y- components form by the following equations:

$$R_x = F_{x1} + F_{x2} + F_{x3} + ... \quad \text{and}$$

$$R_y = F_{y1} + F_{y2} + F_{y3} + ...$$

The magnitude $R$ of the resultant then is

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

and its direction with respect to 'positive-X' axis is given by

$$\tan\Theta = \frac{R_y}{R_x}; \quad \Theta = \tan^{-1}\frac{R_y}{R_x}$$

In the above procedure, the vector components in the 'plus-X' and 'plus-Y' directions have a positive algebraic sign while the components in the 'minus-X' and 'minus-Y' directions have a negative algebraic sign. The angle $\Theta$ of the resultant vector $\mathbf{R}$ is unambiguously determined by considering individually the algebraic signs of $R_x$ and $R_y$.

2. FORCE TABLE

A force table is used to simulate and verify the first condition of equilibrium i.e. an object is in equilibrium only if the vector sum of all the forces acting on it is zero. If all forces are resolved into components, the statement of the first condition of equilibrium can be expressed as

$$\sum F_{xn} = 0, \sum F_{yn} = 0 \quad \text{and} \quad \sum F_{zn} = 0$$

where the summation sign $\sum$ above represents the equation, such as

$$F_{x1} + F_{x2} + F_{x3} + ... + F_{xn} = 0 \quad \text{and}$$

$F_x, F_y$ and $F_z$ are the components of some force $\mathbf{F}$ along $X, Y$ and $Z$-axis, respectively. Figure 5 is a typical representation of a force table showing the ring in equilibrium when four forces are applied using the proper amount of hanging masses on each hanger. One must note that the tension in each string is equal to the weight of the hanging mass(es) (including the mass of the hangar) i.e. the tension $\mathbf{F} = Mg$ where $M$ is the total mass and $g$ is acceleration due to gravity. In practice, the magnitude of the hanging masses can be changed and positions of the pulleys can be adjusted around the force table disk to keep the ring in equilibrium.
3. PROCEDURE

Remove the aluminum post from the center of the table and replace with the bubble level. Level the table with the bubble level by means of the adjustment on the legs. Replace the aluminum post when the table is level. It will be important for all the problems, described below, that the ring is perfectly centered before the data is recorded. Verify that the table is level by placing the ring over the metal post and hanging equal masses over the pulleys every 90°. The ring should stay centered about the post. Gently tapping of the ring will help reduce the frictional effects in the pulleys positions and the hanging masses will help obtain a good set of data. While recording the data, DO NOT FORGET TO INCLUDE THE MASSES OF THE HANGERS AND RECORD ALL DATA AS FORCES.

- **Problem #1**: Set up a force \( F_1 \) at 0° and \( F_2 \) at 90° on the disk. Let the magnitude of \( F_1 \) be about one-half the magnitude of \( F_2 \). Carefully [experimentally] determine the third force \( F_3 \) that causes the system to be in equilibrium.

- **Problem #2**: Set up two forces \( F_1 \) and \( F_2 \) (\(|F_1|=|F_2|\)) such that the angle between them is not 90°. Experimentally determine the third force \( F_3 \) that causes the system to be in equilibrium.

- **Problem #3**: Set up three forces \( F_1 \), \( F_2 \) and \( F_3 \), such that no two forces are at right angle and are NOT the same as the example below. Carefully [experimentally] determine the fourth force \( F_4 \) that causes the system to be in equilibrium.

Record the data from each problem in the Table. Remember mass is a scalar quantity and in order to determine each tension \( \{F_1, F_2, F_3, F_4\} \) you have to multiply 'g' in the appropriate units.

4. EXAMPLE (do not duplicate these specific quantities)

Consider the following experimental data for the ring in equilibrium on the force table.

<table>
<thead>
<tr>
<th>Force [Newtons]</th>
<th>Direction [Degree]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 ) 100 g at = 0°</td>
<td>0.980 0</td>
</tr>
<tr>
<td>( F_2 ) 70 g at = 45°</td>
<td>0.686 45</td>
</tr>
<tr>
<td>( F_3 ) 100 g at = 120°</td>
<td>0.980 120</td>
</tr>
<tr>
<td>( F_4 ) 170 g at = 235°</td>
<td>1.667 235</td>
</tr>
</tbody>
</table>

4.1 GRAPHICAL REPRESENTATION: The forces shown are calculated using \( g = 9.8\text{m/s}^2 \). However, since 'g' is a common multiplying factor, one can make the graphical representation by using the masses only without interfering with the objective of the experiment. We select a scale of 50 g = 2.0 cm and then the graphical representation is shown in Fig. 6. It is to be noted that the fourth vector \( F_4 \) does not end at the beginning point P and therefore one can account for the CLOSURE ERROR by measuring how far the vector \( F_4 \) extends beyond point P, or how short the vector \( F_4 \) is in order to reach point P. In this example, the closure error is about 1/8 inch which corresponds to a mass of about 6.0 g or a force of about 0.06N. The closure error in Fig. 6 results due to error in measurements as well as due to inaccuracies in drawing the vectors in the force diagram. It should also be realized that the balancing force \( F_4 \) in the data is not the resultant \( \mathbf{R} \) of the other three forces (\( \mathbf{F}_1, \mathbf{F}_2, \) and \( \mathbf{F}_3 \)) but known as the EQUILIBRANT \( \mathbf{E} \) or the force that balances the other forces to hold the ring in equilibrium. The EQUILIBRANT is the vector force...
of equal magnitude, but in the opposite direction to that of the resultant i.e. \( \mathbf{R} = -\mathbf{E} \). Therefore, in the graphical representation of forces (Fig. 6), \( \mathbf{F}_4 \) makes an angle of about 46°, with respect to 'plus X' axis and is the EQUILIBRANT.

4.2 ANALYTICAL SOLUTION: Using the data given above, we first find the vector components as follows:

<table>
<thead>
<tr>
<th>Force</th>
<th>X-component(N)</th>
<th>Y-component(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{F}_1 )</td>
<td>0.980</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbf{F}_2 )</td>
<td>0.485</td>
<td>0.485</td>
</tr>
<tr>
<td>( \mathbf{F}_3 )</td>
<td>-0.490</td>
<td>0.849</td>
</tr>
<tr>
<td>( \mathbf{F}_4 )</td>
<td>-0.956</td>
<td>-1.365</td>
</tr>
</tbody>
</table>

Therefore, \( \sum F_x = 0.019N \) and \( \sum F_y = -0.031N \). Ideally, for the first condition of equilibrium to be satisfied, both summations must be equal to zero. Alternatively, one can resolve any three of these four forces into their components and find the resultant \( \mathbf{R} \) of these forces. The fourth force will then be the EQUILIBRANT. For example, we find the resultant \( \mathbf{R} \) of three forces \( \mathbf{F}_1, \mathbf{F}_2, \) and \( \mathbf{F}_3 \) as follows:

\[
\begin{align*}
R_x &= F_{1x} + F_{2x} + F_{3x} = 0.975N \\
R_y &= F_{1y} + F_{2y} + F_{3y} = 1.334N \\
\end{align*}
\]

therefore,

\[
R^2 = (R_x)^2 + (R_y)^2 \quad \text{i.e.} \quad R = 1.652N
\]

and

\[
\tan \Theta = \frac{R_y}{R_x} \quad \text{i.e.} \quad 53.84° \approx 54°
\]

Obviously, \( \mathbf{R} \) is almost equal in magnitude and opposite in direction with respect to the fourth force \( \mathbf{F}_4 \) (equilibrant). The error in magnitude is 1.667 - 1.652 = 0.015N and in the direction it is 235° - (180° + 54°) = 1°.

5. QUESTIONS and WORK TO SUBMIT

(1) A clearly labeled graphical representation for each problem, as in Fig. 6. Measure and clearly indicate the closure error on your plots. Remember to indicate the scale you have used in representing the force magnitudes.

(2) For problems #1 and #2, use the analytical method to determine the resultant \( \mathbf{R} \) of forces \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \). How does the magnitude and direction of \( \mathbf{R} \) compare with \( \mathbf{F}_3 \) [i.e.: calculate the percent difference of the measured resultant (\( \mathbf{F}_3 \)) and your calculated value: (\( \mathbf{F}_{3\text{meas}} - \mathbf{F}_{3\text{scale}} \)) / \( \mathbf{F}_{3\text{scale}} \) *100%].

(3) For problem #3, use the analytical method to determine the resultant \( \mathbf{R} \) of forces \( \mathbf{F}_1, \mathbf{F}_2, \) and \( \mathbf{F}_3 \). How does the magnitude and direction of \( \mathbf{R} \) compare with \( \mathbf{F}_4 \) [i.e.: calculate the percent difference of the measured resultant (\( \mathbf{F}_4 \)) and your calculated value: (\( \mathbf{F}_{4\text{meas}} - \mathbf{F}_{4\text{scale}} \)) / \( \mathbf{F}_{4\text{scale}} \) *100%]. Neatly show all of your work for at least one problem.

(4) What is the physical meaning of the closure error?

(5) What is the physical meaning of the difference between calculated and measured values for the resultant force?

(6) Discuss the possible sources of error within the experiment that could account for the discrepancies in analytical and graphical results. Note: "human error" is NOT acceptable as a source. Be specific about what the source(s) are and how they may affect the results.
GRAPHICAL REPRESENTATION OF VECTORS
(as discussed in the text)

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6