9.53 Using $D(E) = (2s+1)/\hbar\omega_0$,

and

$$\frac{N\hbar\omega_0}{2s+1} = \mathcal{E}$$

and

$$N = \int_0^\infty \mathcal{N}(E)\, D(E)\, dE,$$

solve for B in 9-32, and 9-33.

Then, show for a system of simple harmonic oscillators,

$$\mathcal{N}(E)_{BE} = \frac{1}{\dfrac{e^{E/k_BT}}{1-e^{-\mathcal{E}/k_BT}} - 1}$$

and

$$\mathcal{N}(E)_{FD} = \frac{1}{\dfrac{e^{E/k_BT}}{1-e^{-\mathcal{E}/k_BT}} + 1}$$

Bose-Einstein:

$$\frac{1}{B\, e^{E/k_BT} - 1} = \mathcal{N}(E)$$

Fermi-Dirac

$$\frac{1}{B e^{E/k_BT} + 1} = \mathcal{N}(E)$$

$D(E) \neq f(E)$, so we can pull it out of the integral.

$$N = \left(\frac{2s+1}{\hbar\omega_0}\right) \int_0^\infty \frac{1}{Be^{E/k_BT} \pm 1}\, dE$$

$$z = E/k_BT$$

$$dE = k_BT\,dz, \text{ so}$$

$$\frac{\hbar\omega_0 N}{(2s+1)k_BT} = \int_0^\infty \frac{dz}{Be^z \pm 1}$$

$$\frac{\varepsilon}{k_BT} = \pm \ln\left(1 \pm \frac{1}{B}\right)$$

using the integral given in the problem on the right and the definition of ε on the left.

$$\exp\left(\pm \frac{\varepsilon}{k_BT}\right) = 1 \pm \frac{1}{B}$$

$$\frac{1}{B} = \mp 1 \pm \exp\left(\pm \frac{\varepsilon}{k_BT}\right)$$

$$B = \frac{1}{\mp 1 \pm \exp(\pm \varepsilon/k_BT)}$$

$$B = \frac{1}{1 - \exp\left(\frac{-\varepsilon}{k_BT}\right)} \quad \text{for } \cancel{\text{Fermi-Dirac}} \text{ Bose-Einstein}$$

(i.e. taking the bottom symbols)

Taking the top symbols gives

$$B_{FD} = \frac{1}{e^{\varepsilon/k_BT} - 1}$$

We can then go back to the original equations and substitute in for B to get the full $N(\varepsilon)$ expressions.

(9.64)

Determine the density of states for a 2D infinite well (ignoring spin) in which

$$E_{n_x, n_y} = (n_x^2 + n_y^2)\frac{\pi^2\hbar^2}{2mL^2}$$

We can follow the 3D example from the textbook fairly closely.

$$E = \frac{\pi^2\hbar^2}{2mL^2}n^2, \text{ where } n = \sqrt{n_x^2 + n_y^2}$$

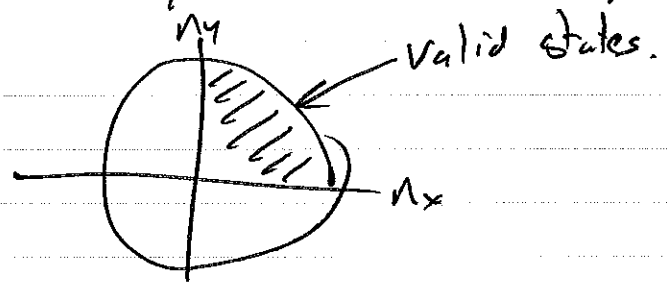
$$n = \sqrt{\frac{2mL^2E}{\pi^2\hbar^2}}$$

Now, instead of a "sphere" working out in 3D space, we have a "circle" in 2D space.

differential # of states in dE =

$$\left(\frac{1}{4}\right)(2\pi)n\,dn$$

The 1/4 is because 1/4 of the states are in a quadrant with positive n_x, n_y



Then,

$$D(E) = \frac{1}{4} 2\pi n \frac{dn}{dE}$$

$$= \frac{\pi}{2} n \frac{dn}{dE}$$

Now, $dE = \left(\frac{\pi^2 \hbar^2}{\cancel{2} m L^2}\right)(\cancel{2} n \, dn)$

So $\frac{dn}{dE} = \frac{m L^2}{\pi^2 \hbar^2 n}$

Then,

$$D(E) = \left(\frac{\cancel{\pi}}{2} \cancel{n}\right)\left(\frac{m L^2}{\pi^{\cancel{2}} \hbar^2 \cancel{n}}\right)$$

$$\boxed{D(E) = \frac{m L^2}{2\pi \hbar^2}}$$

Thus, in the 2D case, the density of states in not a function of energy.

9.74. (a) Show that the number of photons per unit volume in a photon gas of temp. T is approx. $2\times10^{7}\,K^{-3}m^{-3}(T^3)$

Note: $\int_0^\infty x^2(e^x-1)^{-1}dx \simeq 2.4$

Let's start with:

$$dU_{photon} = \frac{hf^3}{e^{hf/k_BT}-1}\,\frac{8\pi V}{c^3}df$$

$$dN_{photon} = (hf)^{-1}\,dU_{photon}.$$

$$dn_{photon} = (dN_{photon})(V)^{-1}.$$

Then,

$$dn_{photon} = \frac{f^2}{e^{hf/k_BT}-1}\,\frac{8\pi}{c^3}df$$

Now, let $x = \frac{hf}{k_BT}$

$$dx = \frac{h}{k_BT}df \Rightarrow df = \frac{k_BT}{h}dx$$

$$f = \frac{k_BT}{h}x$$

$$f^2 = \left(\frac{k_BT}{h}\right)^2 x^2$$

$$dn_{photon} = \left(\frac{8\pi}{c^3}\right)\left(\frac{k_B T}{h}\right)^3 \frac{x^2}{e^x - 1}\,dx$$

$$n_{photon} = \left(\frac{8\pi}{c^3}\right)\left(\frac{k_B T}{h}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\,dx$$

$$= (2.4)(8\pi)\left(\frac{k_B T}{hc}\right)^3$$

Substituting in the numbers gives

$$\boxed{n_{ph} = (2 \times 10^7\ K^{-3} m^{-3})\, T^3}$$

(b) $$U_{ph} = \frac{8\pi^5 V k_B^4 T^4}{15(hc)^3}$$

Then, $$\frac{U_{ph}}{V n_{ph}} = \frac{\cancel{8}\pi^{\cancel{5}\,4}(k_B T)^{\cancel{4}} \cancel{(hc)^3}}{(2.4)\cancel{(8\pi)}\cancel{(k_B T)^3}\cancel{(hc)^3}(15)}$$

$$\frac{U_{ph}}{N_{ph}} = \frac{\pi^4}{(2.4)(15)} k_B T$$

$$\simeq \boxed{2.7\, k_B T}$$

9.50 The laser's resonant cavity must have a half-integer number of wavelengths.

The wavelength range is

~~629.999 to 6~~

632.999 to 633.001 nm

To fit a half integer number of wavelengths,

$\frac{120\text{ cm}}{\lambda}$ must be an integer.

$$\frac{120\text{ cm}}{632.999\text{ nm}} = 1895737.6$$

$$\frac{120\text{ cm}}{633.001\text{ nm}} = 1895731.6$$

So, the number of half-wavelengths can be 1895732 − 7, so there are 6 of these.

(b) To get a single wavelength out, one must use a shorter cavity.