

Week 6: Stability of stars

In this week, we will discuss how we determine whether stars are stable, and we will consider some key instabilities that have strong effects on the evolution of stars.

There are three kinds of equilibria in nature – stable, unstable, and saddle points. One can see which type of equilibrium one has by plotting the key quantities versus one another (e.g. energy versus displacement). Stable equilibria are local minima in energy, unstable equilibria are local maxima in energy, and saddle points are positions where there is a zero first derivative, but neither a minimum nor a maximum. Saddle points can exist most readily in multiple dimensions, where the position represents a minimum with respect to one direction and a maximum with respect to another, but they can also exist for functions like $y = x^3$.

We need to worry about both thermal stability and hydrostatic stability.

First, let's consider whether typical stars are globally stable under normal circumstances. We can write down several different approaches to looking at the virial theorem. For a gas pressure dominated star:

$$3 \int_0^M \frac{P}{\rho} dm = -\Omega U = -\frac{1}{2}\Omega E = \frac{1}{2}\Omega = -U \quad (1)$$

With radiation pressure added:

$$\frac{P}{\rho} = \frac{P_{gas}}{\rho} + \frac{P_{rad}}{\rho} = \frac{R_{Gas}T}{\mu} + \frac{aT^4}{3\rho} = \frac{2}{3}u_{gas} + \frac{1}{3}u_{rad} \quad (2)$$

where R_{gas} is the gas constant written as script R in class and in the book, and the other quantities have the usual meanings.

Now, via the virial theorem,

$$u_{gas} = -\frac{1}{2}(u_{rad} + \Omega)E = -U_{gas} \quad (3)$$

Now, remember that Ω is a negative number. Thus, the effect of radiation pressure is to reduce the effective gravitational attraction.

Next, let's consider that $\dot{E} = L_{nuc} - L$ - i.e. the rate of change of energy is equal to the energy input from nuclear power minus the energy output from radiation. Thermal equilibrium occurs for $L_{nuc} = L$.

What happens if we perturb the star, so that L_{nuc} temporarily exceeds L ? Then $\dot{E} > 0$, and since $E < 0$, $|E|$ will be reduced. Then, since $U = -E$, U drops, the star will cool, and the nuclear reaction rate will drop. We call this *secular stability* – that the overall long term evolution of the star cannot be dramatically affected by a small perturbation.

Thermal instability

Stability will be ensured when a perturbation affects the energy supply in the opposite manner of the perturbation – i.e. as above, when increasing the energy production rate will cause a change that reduces the energy production rate.

Degenerate gases can be thermally unstable. For degenerate gases, the pressure is not a function of temperature (or at least it's a very weak function of pressure). $E = -U$ still holds, so expansion will still take place if an increase in the nuclear luminosity happens, but a drop in temperature will not result. Nuclear reaction rates are much more sensitive to temperature than to density, so the reaction rate will not be strongly affected by the expansion, and the temperature will rise, so the reaction rate will then go up. We get runaway nuclear burning.

Two possibilities exist: a total runaway that leads to an explosion, or a situation where the gas is heated to the point that gas pressure exceeds degeneracy pressure, and the gas then cools as it expands.

Some examples of runaways are classical novae (in a small layer on the surface of a white dwarf), Type I X-ray bursts (in a small layer on the surface of neutron star) and Type Ia supernovae (globally throughout an entire white dwarf).

Now we can consider how to write down a general criterion for thermal stability. For stars burning nuclear fuel at their centers, recall that for all polytropic models,

$$\frac{dP_c}{P_c} = \frac{4}{3} \frac{d\rho_c}{\rho_c}. \quad (4)$$

It will also then be true that:

$$\frac{dP_c}{P_c} = a \frac{d\rho_c}{\rho_c} + b \frac{dT_c}{T_c}, \quad (5)$$

setting this up as a gas law. Then,

$$\left(\frac{4}{3} - a\right) \frac{d\rho_c}{\rho_c} = b \frac{dT_c}{T_c} \quad (6)$$

For $a < \frac{4}{3}$, the two sides will have the same sign, and hence contraction will cause heating and expansion will cause cooling. Heating will drive expansion by increasing the total energy content of the gas, and this expansion will then lead to cooling. The stars are thus stable in this regime. For ideal gases, $a = 1$, so we are in this regime.

For degenerate gases, $a \geq \frac{4}{3}$, and a rise in the energy input will lead to slight heating, causing a runaway unless/until the heating makes the gas become ideal.

Thin shell instability

Now, let's consider the case of a thin shell in a star with properties very different from those of the gas on either side. This can develop due to shell burning (i.e. cases where there are thin shells where a particular element is undergoing nuclear fusion) in the late stages of stellar evolution, so it's not a contrived situation.

We'll let this shell have mass Δm , and temperature T , and be located in a star with radius R , at height r_0 , and have thickness ℓ .

The pressure gradient in the star is given by $\frac{dP}{dm} \propto r^{-4}$, so $\frac{dP}{P} = -4 \frac{dr}{r}$. We also have $\Delta m = 4\pi r_0^2 \ell \rho$. We can derive that $\frac{d\rho}{\rho} = -\frac{dr}{r} \frac{r}{\ell}$. We can then

substitute in the pressure gradient and get:

$$\frac{dP}{P} = 4 \frac{\ell}{r} \frac{d\rho}{\rho}. \quad (7)$$

Making the same substitution of a general equation of state for the gas in as we did above, we get:

$$\left(4 \frac{\ell}{r} - a\right) \frac{d\rho}{\rho} = b \frac{dT}{T}, \quad (8)$$

so $a < 4 \frac{\ell}{r}$ is needed for stability. Essentially, for very thin shells, the pressure from the rest of the star is so large compared to the internal pressure that changes in the internal pressure don't lead to substantial expansion.

Dynamical instability

Next, let's consider how to test whether a star is dynamically unstable – that is whether a small perturbation in radius will lead to a restoring force or a runaway.

Consider a sphere of mass M , with $P(M) = 0$.

Then:

$$P = \int_M^m \frac{Gm' dm'}{4\pi r^4} \quad (9)$$

gives the pressure at the radius containing mass m .

The density at $r(m)$ is given by $\rho = \frac{1}{4\pi r^2} \frac{dm}{dr}$.

Now consider what happens if the star is compressed uniformly, so that $r' = (1 - \epsilon)r$. If $\epsilon \ll 1$, then we can use the first term of the binomial expansion, and see that $(1 \pm \epsilon)^n = 1 \pm n\epsilon$, so $\rho' = \frac{1}{4\pi r'^2} \frac{dm}{dr'} = \frac{\rho}{1 - \epsilon} = \rho(1 + \epsilon)$.

Next, we can use the gas law to find the gas pressure:

$$P'_{gas} = P(1 + 3\epsilon)^{\gamma_a} = P(1 + 3\epsilon\gamma_a), \quad (10)$$

and the new hydrostatic pressure will be $P'_h = P_h(1 + 4\epsilon)$.

To have stability requires that the new gas pressure is larger than the new hydrostatic pressure, since then a restoring force will bring the star back into equilibrium. This is equivalent to requiring $\gamma_a > \frac{4}{3}$. In some cases there may be small regions in the outer part of the star where $\gamma_a < \frac{4}{3}$ because of ionization changes, and the star will pulsate.

On the other hand, where the global value of $\gamma_a < \frac{4}{3}$, then the whole star will be unstable. This may happen in three cases: relativistic degenerate gases, where the dynamical instability leads to a Type Ia supernova, as the star collapses, triggering runaway nuclear burning; radiation pressure dominated stars, where the star is approaching being unbound as the radiation pressure starts to dominate over gas pressure; and cases where there is an ionization instability (or other instability that leads to sharp changes in the number of particles) in the core of the star, which may happen if the core is dominated by electron-positron pairs or iron disintegration starts taking place.

Convection

The maximum possible radiative flux is given by $\kappa F < 4\pi cGm$ – if this criterion is violated, then hydrostatic equilibrium is no longer possible. For large F (as in the interiors of massive stars) or large κ , as in the envelopes of low mass stars, this criterion can be violated, leading to a loss of local hydrostatic equilibrium, and, instead, convective energy transport. In fact, it’s a bit easier than this to get convection started.

The standard criterion for convection is called the Schwarzschild criterion, after Karl Schwarzschild, who derived it in 1906.¹ A more advanced criterion, the Ledoux criterion, can be derived by considering also the effects of having a chemical composition gradient in the gas. You should be aware that this difference exists, and is sometimes important, but we won’t deal with that subtlety in this course.

I’ll just outline the physics of the derivation here, and give the answer, rather than repeating the entire derivation. We consider a fluid element that is displaced upwards in the star. It is then hotter than the nearby gas, so it will expand until it is in pressure equilibrium. We then consider its density relative to the density of the nearby gas. No heat exchange will be, since the dynamical time is much less than the thermal time, but some work will be done. Now, we compare the density of the fluid element that was displaced with the density of the local gas. If the fluid element is denser than the rest of the star, it will sink back down. Otherwise, it continues to rise due to buoyancy. In the latter case, the convective instability takes place.

The result of a fair bit of algebra is that:

$$\kappa F < 4\pi cGm \left[4 \frac{\gamma_a - 1}{\gamma_a} (1 - \beta) \right] \quad (11)$$

is the criterion which must be satisfied for a star to be stable against convection. This is slightly stricter than the Eddington limit, because the term in square brackets will always be less than 1 – but as β approaches 0 (i.e. the fully radiation pressure dominated case), then the two limits converge (since γ_a is 4/3 for radiation pressure domination).

Thus, where κ is large, in stellar atmospheres of cool stars, for example, convection can set in at low luminosities – in fact stars less than about $0.25 M_\odot$ are convective throughout. Stars with the CNO cycles in their cores are also always convective in the core. A small range of masses exists where small core and envelope convection zones can exist simultaneously.

Convective energy transport: the mixing length

We can next consider how far a mass element can travel before mixing in with its surroundings, and stopping to travel. Even in detailed stellar models, this mixing length parameter is often used as a free parameter because convection is so difficult to calculate directly. We define a number α to be the ratio of the convective distance travelled to the pressure scale height.

¹This is the same Schwarzschild who derived black holes in general relativity – something he did from a trench during WW I. His son, Martin, did the first serious computer models for stars.

We then wish to express the heat transfer in terms of α and other quantities we already know. The δT for a mass element will be: $(|\frac{dT}{dr}|_{star} - \frac{dT}{dr}|_a)\ell_c$, where the first term in parentheses is due to the global temperature gradient, and the second is the adiabatic change of the fluid element moving through the star, and ℓ_c is the distance travelled by the fluid element.

The convective heat flux – energy per unit area, per unit time, will be:

$$H_c = \rho v_c c_p \delta T, \quad (12)$$

with v_c the velocity of the convective cell, equal to $\sqrt{g'\ell_c}$, with g' equal to the buoyant acceleration.

Then, $v_c = \sqrt{\alpha(\frac{P}{\rho})|\frac{\delta T}{T}|}$, and since $\sqrt{P/\rho}$ is the sound speed, there is a simple relation between v_c and the sound speed.

We can then re-write the convective heat flux as:

$$H_c = \rho c_p T \sqrt{P/\rho} [(\frac{d\ln T}{d\ln P})_{star} - (\frac{d\ln T}{d\ln P})_a]^{3/2} \alpha^2 \quad (13)$$

Then, F is equal to the sum of $4\pi R^2 H_c$ and the radiative term, and the temperature gradient can be both large and complicated in form.

Adiabaticity

We can now look at the energy transport and see how it deviates from pure adiabaticity. $\delta T/T$ is the superadiabaticity – the energy transferred. We'll just express things in dimensional units, ignoring small factors.

Then, $H_c = L/R^2$, $\rho = M/R^3$, $c_p T = U/M$, and $P/\rho = GM/R$, and we'll take $\alpha = 1$. Then, $(\delta T/T)^{3/2} \sim L/U(GM/R^3)^{-1/2}$. But the L/U is just the reciprocal of the thermal timescale, and the other term is the dynamical timescale, so we have that the fractional temperature change is $(t_{th}/t_{dyn})^{-2/3}$, which is about 10^{-8} for typical stars. Close to the stellar surface, these approximations may break down, but basically, this small energy change doesn't lead stellar interiors away from being adiabatic, and this is why polytropic models work well.