Week 5: Simple equilibrium configurations for stars

Follows on closely from chapter 5 of Prialnik.

Having understood gas, radiation and nuclear physics at the level necessary to have an understanding of the basic processes inside stars, we can now move to developing some formalisms for the basic equilibrium configurations of stars.

There are four basic equations that need to be solved. Here we will have already made one simplifying assumption, that the chemical composition of the star is constant throughout – this is a good approximation for getting a basic picture of what the main sequence structure of a star looks like, but obviously if this approximation is taken too far, then the star cannot evolve.

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \tag{1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{2}$$

$$\frac{dT}{dr} = \frac{-3}{4ac} \frac{\kappa\rho}{T^3} \frac{F}{4\pi r^2} \tag{3}$$

$$\frac{dF}{dr} = 4\pi r^2 \rho q \tag{4}$$

To these, we add a few relations that describe the pressure, opacity, and heat generation:

$$P = \frac{R}{\mu_I}\rho T + P_e + \frac{1}{3}aT^4 \tag{5}$$

$$\kappa = \kappa_0 \rho^a T^b \tag{6}$$

$$q = q_0 \rho^m T^n \tag{7}$$

Additionally, since we have four first order differential equations, we need four boundary conditions. Three are obvious: at the center, $m = 0$ and $F = 0$, and at the surface, $P = 0$. (Note that this one is only an approximation, depending on how the surface is defined.) The fourth boundary condition must be some relation between the temperature at the surface and the temperature somewhere in the stellar interior.

One can see that this will not be an easy set of equations to solve, and that a numerical approach will be required. The equations are coupled and nonlinear, and the value of pressure varies by 10^{11} between the surface and center, and the value of temperature varies by more than a factor of 1000. Despite all this, considerable progress was made on understanding stellar structure long before the development of electronic computers[1], and before the discovery of the key bits of nuclear physics that allow us to understand energy generation in stars.

What allowed progress to be made was the development of some key simplifications which turn out to be reasonable approximations of what really happens in stars. We have already discussed one – the assumption of uniform composition.

1 Polytropes

One simplifying assumption that can be made is that the equation of state in the star is constant. This is sometimes a very good assumption, and sometimes poor – but it allows for a straightforward calculation of the structure of something like a star.

We can start with the equation 1, multiply by r^2/ρ, and differentiate with respect to r:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}, \tag{8}$$

and we get an equation whose right side can be substituted with $-4\pi G\rho$.

Now, we set $P = K\rho^\gamma$, a "polytropic equation of state", defined by K and n, the polytropic index, which is defined such that $\gamma = 1 + \frac{1}{n}$.

Then, substituting in for P with $K\rho^{1+\frac{1}{1}}$ everywhere, we get:

$$\frac{(n+1)K}{4\pi Gn}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) \tag{9}$$

We then set up several substitutions to non-dimensionalize the equations:

$$\rho = \rho_c\theta^n \tag{10}$$

where ρ_c is the central density;

$$\left[\frac{(n+1)K}{4\pi G{\rho_c}^{\frac{n-1}{n}}}\right] = \alpha^2 \tag{11}$$

and

$$r = \alpha\xi \tag{12}$$

which we can then combine to produce the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{13}$$

[1]Edward Charles Pickering, director of the Harvard Observatory around the turn of the 20th century, had a group of women who worked under him who did calculations by hand, and also classifying spectra of stars, and were known as the "Harvard Computers". A few, notably Annie Jump Cannon, made major contributions to astronomy that went well beyond just doing calculations and classifying spectra.

This equation has two boundary conditions: $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. This still requires numerical integration unless $n = 0$ or $n = 1$, but the numerical integration is fairly straightforward. When $\xi = 0$, the value is ξ is referred to as ξ_1, and the surface of the star has been reached: $R = \alpha\xi_1$.

The mass of the polytrope is given as:

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi \tag{14}$$

Then, $M = 4\pi\alpha^3\rho_c\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$, based on substituting in θ^n from the original Lane-Emden equation.

For $n = 0$ and $n = 1$, there are simple analytic solutions to the Lane-Emden equation, but in general, numerical calculations are needed – but, unlike for the case of the full set of stellar evolution equations, these numerical calculations can be done in a straightforward way, since we now have transformed the problem into integration of a simple second order differential equation in one variable.

We can define 4 polytropic constants – D_n, M_N, R_N, and B_n, where the first three are easy to remember, since they relate to density, mass, and radius, and the fourth is a number close to 1 that serves as a catch-all in the final pressure equation.

$$\rho_c = D_n\bar{\rho} \tag{15}$$

defines D_n, which is equal to $-\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)\right]^{-1}$.

Next, we can take the mass equation, and the defintion of α, and find:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}, \tag{16}$$

where $M_n = \xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$ and $R_n = \xi_1$. We note that this gives a mass-radius relation for a given polytropic index.

For n=3, the mass is not a function of the radius – it's only a function of K, the scale factor for the pressure-density relationship:

$$M = 4\pi M_3\left(\frac{K}{\pi G}\right)^{3/2} \tag{17}$$

Thus, for a given K there is only a single possible value of mass for an $n = 3$ polytrope.

Another special case is the $n = 1$ polytrope, in which only one value of R can exist, $R = R_1\left(\frac{K}{2\pi G}\right)^{1/2}$ and it is independent of M.

In between,

$$R^{3-n} \propto \frac{1}{M^{n-1}}, \tag{18}$$

meaning that more massive stars are denser.

Now, we can pull out K from the mass-radius relation, and use $P_c = k\rho_c^{1+\frac{1}{n}}$, and put all the remaining n dependence into B_n, which turns out always to be approximately 1, and we can get:

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}, \tag{19}$$

yielding the result that $P_c \propto M^{2/3}\rho_c^{4/3}$ for nearly any equation of state.

2 The Chandrasekhar mass

Despite the fact that white dwarfs are exotic stars, and inherently quantum mechanical, they actually provide one of the *easiest* challenges for stellar physics. This is because white dwarfs are the one class of stars with very simple equations of state.

Let's first consider the special case where the white dwarf is low mass, so purely non-relativistic, and has a sufficiently low temperature that thermal motions can be ignored. It then follows an $n = 1.5$ polytrope, since its equation of state is exactly that for a non-relativistic degenerate electron gas. Then, $R \propto M^{1/3}$, and $bar\rho \propto MR^{-3} \propto M^{-2}$. Eventually, the density becomes so large that the electrons must become relativistic. Then, the solution will approach a $n = 3$ polytrope, for which only one mass is possible (for a given value of K). We can then take the value of K for degenerate relativistic electrons from a few weeks/chapters back in the notes/book, and find:

$$M_{ch} = \frac{M_3\sqrt{1.5}}{4\pi}\left(\frac{hc}{Gm_H^{4/3}}\right)^{3/2}\mu_e^{-2}, \tag{20}$$

which yields $M_{ch} = 1.46M_\odot$ for matter with equal numbers of neutrons and protons. For iron, the number will be about 15% lower. For pure hydrogen, it would be a factor of 4 higher, but of course in a star made of $5M_\odot$ of hydrogen, fusion would take place, so the point is moot.

3 The Eddington Luminosity

Is there some fundamental limit to the rate at which energy can be produced in a star, or the rate at which energy can be transported through a star, without a violation of the basic assumptions that go into what we've done here? The answer is yes; although for the latter question, we can consider alternative means of energy transport.

So, let's start with the equation for radiation pressure: $P_{rad} = \frac{1}{3}aT^4$. We can then differentiate this with respect to radius; substitute into equation 3; and divide by equation 1. This yields: $\frac{dP_{rad}}{dP} = \frac{\kappa F}{4\pi cGm}$.

This gives an upper limit for the stellar luminosity and for the star to be in hydrostatic equilibrium, since $\frac{dP_{rad}}{dP} < 1$ (since the radiation pressure must everywhere be less than the total pressure (or the gas pressure would be negative), and hence cannot change by more than the total pressure). Then, $\kappa F < 4\pi Gcm$, and $L < \frac{4\pi Gcm}{\kappa}$.

Now, at high heat flux, or at low opacity, this condition may be violated, and then we lost hydrostatic equilibrium, at least locally. In the interior of a star, convection will occur, and in the exterior, a wind will be driven. We also get a local maximum rate of energy generation at the center of the star, $q_c < \frac{4\pi cG}{\kappa}$.

The maximum luminosity, called the *Eddington luminosity* can be re-written as:

$$L_{EDD} = 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right)\left(\frac{\kappa}{\kappa_{es}}\right)^{-1} L_\odot \tag{21}$$

Since the solar luminosity is about 4×10^{26}W, or 4×10^{33}ergs/sec, the Eddington luminosity can also be written as 1.38×10^{31}W or 1.38×10^{38}ergs/sec.

4 Eddington's "standard" model

This treatment of radiation pressure as a fundamental quantity can help motivate an approach to a key historical simplifcation of the equations of stellar physics, one also developed by Eddington. Eddington found that with the assumption that the total pressure is proportional to gas pressure everywhere in the star, then an $n = 3$ polytrope can be taken as the solution for stellar structure in all cases. This leads to mass being a function of K, with K a function of the ratio of radiation to gas pressure.

Eddington justified the assumption that gas pressure is proportional to radiation pressure with an approach that is not rigorous, but which was instrumental in getting some sort of stellar model to work. He first wrote:

$$\frac{F}{m} = \eta \frac{L}{M}, \tag{22}$$

defining η as the ratio between luminosity and mass produced within a radius and the same quantity at the outside of the star. It is clear that η increases inwards in the star.

He next noted that $\frac{dP_{rad}}{dP} = \frac{\kappa F}{4\pi cGm}$, which we see above. Clearly, since the temperature is larger inwards and Kramer's opacity decreases sharply with increasing temperature, κ should increase outwards. Eddington then decided to calculate what would happen if one assumed $\kappa\eta \equiv \kappa_s$, where κ_s is a constant. Then:

$$P_{rad} = \frac{\kappa_s L}{4\pi cGm} P, \tag{23}$$

where $L = L_{EDD}(1-\beta)$, where $\beta = \frac{P_{gas}}{P}$, and we now have constant β.

We can then define $\frac{aT^4}{3(1-\beta)} = \frac{R}{\beta\mu}\rho T$.

This yields three equations:

$$T = \left[\frac{3R(1-\beta}{a\mu\beta} \right]^{1/3} \rho^{1/3}, \tag{24}$$

$$P = K\rho^{4/3}, \tag{25}$$

and,

$$K = \left[\frac{3R^4(1-\beta)}{a\mu^4\beta^4} \right]^{1/3}. \tag{26}$$

Since K is a constant, we get an $n = 3$ polytrope. Then, we get $M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}$, and we can substitute in M_3 and R, and we get Eddington's quartic equation:

$$1 - \beta = 3 \times 10^{-3} \left(\frac{M}{M_\odot} \right)^2 \mu^4 \beta^4. \tag{27}$$

An interesting point is that only in a narrow range of M is β anything other than very close to 0 or very close to 1 – i.e. only for a narrow range of M is the star anything other than purely gas pressure dominated or purely radiation pressure dominated – and this range turns out to be the range for which stars actually exist.

Two key general results can be found:

(1) Massive stars are radiation pressure dominated – to the extent that very massive stars come very close to the Eddington luminosity and probably drive stellar winds (which they are observed to do).

(2) If we take Eddington's quartic equation, and substitude L/L_{Edd} for $(1-\beta)$, then note that $L_{EDD} \propto M$, we find that $L \propto M^3$, which is a pretty good approximation to reality, at least for massive main sequence stars.

This was not known from observations at Eddington's time.

It is fascinating to see how far Eddington came toward describing stellar structure reasonably accurately with essentially no understanding of the nuclear physics that provided the energy source!

5 The Cowling approximation – point source models

An alternative simplification – one which can better describe the lower main sequence – is that all of the nuclear power is generated from a point in the center of the star. Then, we can set $F = L$, and let F be a constant. The solutions are a bit more complicated and require numerical integration, but they were numerical integrations [2] that could be done in the 1930's.

[2] Numerical integration basically consists of plotting a function, then fitting rectangles or trapezoids, or other shapes along the curve, and adding up their areas. Some methods use some tricks to reduce the number of shapes that need to be drawn, but increase the complication of the shape so that it's a closer approximation of the curve, but the basic underlying idea is the same.

Next, we set $\kappa = \kappa_0 \rho^a T^b$, to define the opacity, since it need not be related to the radiation pressure.

Then:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{28}$$

$$\frac{dP_{rad}}{dr} = -\frac{\kappa_0 L}{c} \frac{\rho^{a+1} T^b}{4\pi r^2} \tag{29}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{30}$$

This can be integrated numerically for any given opacity model. For the simplest opacity model, $a = b = 0$, constant opacity (relevant, e.g. in cases where electron scattering dominates), some additional progress can be made in solving the equations in a more straightforward way. We won't go into this level of detail in this course – by we note that the resulting solutions give a steeper $M - L$ relationship than the Eddington model – and thus they agree better with real low mass stars.

This is not surprising – the Eddington model implicitly assumes that fusion is taking place all the way out in the envelope of the star, whereas low mass stars barely have high enough density and temperature to have any fusion at all – putting all the fusion in their cores is a better approximation.