# Week 3: Basics of gas and radiation physics

This week, we will go through the basics of gas and radiation physics, with, not surprisingly, a focus on the aspects that apply in stars. Some of you may have seen at least bits of this material in some other classes (particularly if you've already taken statistical mechanics), but I will approach it as though it's completely new to everyone.

Much of stellar physics concerns itself with trying to understand the *equations of state* of gases. An equation of state is just a relation between pressure and density. The simplest equation of state is the ideal gas law, which gives pressure proportional to density. Now, let's try to justify (or refute) that law as it applies in stars.

We can go back to the calculation we did a few lectures ago, in which we found that the mean temperature of a star is high enough that the gas should be fully ionized. Thus, at least deep in the stellar interior, this should be the case. We thus need to worry only about Couloumb interactions, and not molecular nor atomic processes.

We can then find the mean distance between charged particles by taking the cube root of the number density of particles, and get:

$$d = \left(\frac{4\pi A m_h}{3M}\right)^{\frac{1}{3}} R. \tag{1}$$

The Coulomb energy per particle can be found to be:

$$\epsilon_C \approx \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{Z^2e^2}{d}\right). \tag{2}$$

We can then substitute in the value for $k_B\bar{T}$, the mean temperature, that comes from finding the internal energy by applying the virial theorem and the gravitational binding energy. We get $\epsilon_C/k_bT = 0.01(M/M_\odot)^{-2/3}$. Note that there is no radius dependence here. Since the minimum mass of a star is $0.08M_\odot$, we find that all stars have negligible Couloumb interactions in their cores. On the other hand, for planets, which have masses less than $10^{-3}M_\odot$, the Couloumb interactions can be extremely important – this explains why planets are in the regime where matter makes a transition from gas to solid. [Also - someone asked in class if Jupiter has a solid core - the answer is that as of the 2008 article I could find on the topic, it was not a settled issue].

*Types of pressure and pressure relations*

A generic expression for pressure, which applies both to particles with mass (gas pressure) and light (radiation pressure), can be derived:

$$P = \frac{1}{3}\int_0^\infty vpn(p)dp, \tag{3}$$

where $v$ is velocity, and $p$ is momentum. This relation can be proved by considering the transfer of momentum that takes place when a particle bounces off a wall.

If we consider non-interacting species, the pressures of the different species can be added to get a total pressure. In a star, the species will be ions, electrons, and photons:

$$P = P_I + P_e + P_{rad} = P_{gas} + P_{rad}, \text{and} P_{gas} = \beta P \tag{4}$$

defines $\beta$, the ratio of gas pressure to total pressure, a quantity which is used extensively in Eddington's simple stellar model.

Ion pressure can be computed in a straightforward way. We take the ideal gas law, $P_I = n_I k_B T$, which we can find from integrating equation 3 with a Maxwellian velocity distribution. We can then consider different types of ions:

$$n_I = \sum_i n_i = \sum \frac{\rho}{m_h} \frac{X_i}{A_i}. \tag{5}$$

We can then set a mean atomic mass, $\mu_I \equiv \sum_i \frac{X_i}{A_i}$, allowing us to set $n_I = \frac{\rho}{\mu_I m_H}$.

Then,

$$\frac{1}{\mu_I} \approx X + \frac{Y}{4} + \frac{1 - X - Y}{< A >}, \tag{6}$$

where $X$ is the hydrogen abundance, $Y$ is the helium abundance, and $1 - X - Y = Z$ is the metal abundance. We will generally avoid using $Z$ in this course in cases where there might be some confusion with the ion charge, but be aware that it is common usage in scientific papers.

Now, we can then make another definition of a new symbol: $R \equiv \frac{k_B}{m_H}$, so $P_I = \frac{R}{\mu_I}\rho T$. The textbook uses a fancy script $R$ for this, but I cannot figure out how to make that display in LaTeX.

The value of $mu_I$ for the Sun is about $(0.74 + 0.24/4 + 0.02/20)^{-1}$, so $\mu_I$ is about 1.25.

Electron pressure can be treated in essentially the same way, except that there are many electrons per atom for more highly charged elements:

$$\frac{1}{\mu_e} = X + \frac{1}{2}Y + (1 - X - Y)\left\langle \frac{Z}{A} \right\rangle. \tag{7}$$

For metals, $\left\langle \frac{Z}{A} \right\rangle$ is about 0.5, since most of the ions are in the lower $Z$ metals, which have equal or nearly equal numbers of protons and neutrons.

Then, $\mu_e \approx 2(1+X)^{-1}$, and $\mu_e$ is about 1.17 for solar composition material, and 2 for material with almost no hydrogen (as, for example, white dwarfs will be).

We can then set $P_e = \frac{R}{\mu_e}\rho T$. Summing the two components of the gas pressure together, we get:

$$\frac{1}{\mu} \equiv \frac{1}{\mu_I} + \frac{1}{\mu_e}, \tag{8}$$

and $P_{gas} = P_I + P_e = \frac{R}{\mu}\rho T$, with $\mu = 0.61$ for solar composition material.

Thus, for pure hydrogen, the gas pressure will be half electron and half ion, while for elements with neutrons, the electron pressure will dominate.

We have assumed so far that there is complete ionization and that there are no interactions between particles. These are generally fine, but are purely classical. Degeneracy pressure can often be important in stars:

$$\Delta V \Delta^3 p \geq h^3 \tag{9}$$

is an expression of the uncertainty principle that's in a useful form for us – with $\Delta V$ the volume per particle, and $\Delta p$ momentum, with the cubing because we're working in three dimensions.

If we compress gas, $\Delta V \propto \rho^{-1}$, and eventually, $\Delta V$ becomes small, so that we'll be required to enforce a minimum momentum above the thermal value.

Because proper calculations are difficult, we'll develop intuition by considering two cases: non-relativistic electrons at absolute zero temperature, and ultrarelativistic electrons at absolute zero temperature, but with the speeds purely from degeneracy pressure, and not from random thermal motions.

We can find from combining the Heisenberg and Pauli principles that:

$$n_e = \int_0^{p_0} \frac{2}{h^3} 4\pi p^2 dp. \tag{10}$$

One can then integrate out the electron density in momentum space, and get the electron density, which yields $p_0 = \left(\frac{3h^3 n_3}{8\pi}\right)^{1/3}$. We can then take equation 3 and integrate it with $v = p/m_e$, and we find:

$$P_{e,deg} = \frac{8\pi}{15 m_e h^3} p_0^5 = frach^2 20 m_e \left(\frac{3}{\pi}\right)^{2/3} m_H^{-5/3} \left(\frac{\rho}{\mu_e}\right)^{5/3}. \tag{11}$$

Note that the degeneracy pressure is inversely proportional to the particle mass (the $m_H$ term is just to convert between $\rho$ and particle number density). This makes electron degeneracy pressure relevant at far lower densities than baryon degeneracy pressure. This means also that when we do deal with neutron degeneracy pressure, it will happen under extreme relativistic conditions (i.e. general relativity and not just special relativity becomes important, and it turns out that some effects in particle physics become important, and these are not all well understood). The equation of state for neutron stars, and hence the mass-radius relation for neutron stars, is still a topic of active research, both experimentally and theoretically.

We can also consider the opposite extreme, that of degeneracy of relativistic electrons. Even for white dwarfs, $p_0/m_e$ approaches the speed of light. Fortunately, we are still at low enough densities that we can do with with just special, rather than general relativity. Replacing $v$ by $c$, and $mv$ by $\gamma mc$, and doing the integrals out, we get:

$$P_{e,r-deg} = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_H^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3}. \tag{12}$$

In reality, as the white dwarf mass increases, there will be a gradual transition between the two branches.

We should also remember that there will be stars where degeneracy pressure is a partial contributor to the total pressure, where $k_BT$ is an appreciable, but not large fraction of $p_0$. For such stars, there will be some particles at $p > p_0$, but there will be a skewing of the momentum distribution to higher momenta than a Maxwellian with $k_BT$ would have at infinitessimal density.

*Radiation pressure*

Now we can consider the concept of radiation pressure. When photons transfer momentum to gas particles, they exert radiation pressure. In thermodynamic equilibrium, we get the Planck distribution for photons frequencies:

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{d\nu}{e^{\frac{h\nu}{k_BT}}-1}. \tag{13}$$

Given that $v = c$ and $p = E/c$ for photons, we can integrate this through equation 3, and get:

$$P_{rad} = \frac{1}{3}\int_0^\infty c\frac{h\nu}{c}n(\nu)d\nu = \frac{a}{3}T^4, \tag{14}$$

with $a = \frac{8\pi^5 k_B^4}{15c^3h^3} = \frac{4\sigma}{c}$, where $\sigma$ is the Stefan-Boltzmann constant.

*Internal energy of gas and radiation*

Next, it's useful to look at the internal energies for gas and radiation. The internal energy per unit mass, $u$ can be written as:

$$u = \frac{1}{\rho}\int_0^\infty n(p)\epsilon(p)dp \tag{15}$$

where $\epsilon(p)$ is the energy associated with momentum $p$. We can then integrate over a Maxwellian for particles or over a Planck function for photons. (In principle, we could integrate over non-equilbrium distributions as well, but we will see in a few pages that the equilibrium distributions are excellent approximations of what happens in stellar interiors).

For non-relativistic particles, $\epsilon(p) = \frac{p^2}{2m_g}$, while for relativistic particles, we need to use $(\gamma-1)m_gc^2$, where $\gamma$ here is the Lorentz factor. The "minus 1" term is so that we subtract off the rest mass of the particle, and integrate out only the kinetic energy.

$u_{gas} = \frac{3}{2}\frac{P_{gas}}{\rho}$ is the ideal gas law in a new form, since $u_{gas} = \frac{3}{2}nk_BT/\rho$, and $P_{gas} = nk_BT$. For relativistic degenerate electrons, we can (in principle) run through some algebra, and find that $u_{gas} = 3\frac{P_{gas}}{\rho}$, the same result we get for photons by integrating out the Planck distribution with $\epsilon = h\nu$. Thus we see an example of something that happens frequently in physics – at ultrarelativistic speeds, particles and photons begin to act in very similar ways. There are some connections between this result and grand-unified theories of particle physics, although the grand unified theories are obviously much more involved than just this.

*Adiabatic processes*

Adiabatic processes are ones in which there is no heat exchange with the environment. We can go through a fair bit of algebra and we end up proving that for such processes, if $u = \phi\frac{P}{\rho}$, then $P \propto \rho^{\frac{\phi+1}{\phi}}$. We then define $\gamma_a \equiv \frac{\phi+1}{\phi}$, and call it the adiabatic exponent, and the pressure-density relation is then the equation of state for adiabatic processes in that gas. For non-relativistic ideal gases, this will be $\frac{5}{3}$, while for relativistic gases or photons, it will be $\frac{4}{3}$. In more complicated cases, the value will tend to lower values – if there are internal energy states, there are other modes in which to store energy besides just kinetic energy, and this will increase the value of $\phi$, leaving $\frac{\phi+1}{\phi}$ smaller as $\phi$ increases. Complex molecules, with their large numbers of vibrational and rotational modes, can have $\gamma_a$ approaching 1. In stellar interiors, $\gamma_a$ will be $\frac{5}{3}$ except when dominated by radiation pressure, or when in a region of space where an important atom's ionization state is changing quickly, or photon-photon pair production is taking place. By important atoms, we basically mean $H$ and $He$ except in some evolved stars where metals can dominate the opacity.

*Radiative transfer*

Now we can start to consider how light propagates through a star. We can make the "plane-parallel" approximation for most of the star. Near the surface, treating the star as a series of parallel planes can lead to some problems (e.g. it wouldn't account properly for limb darkening). We won't do much with stellar atmospheres in this course, but bear in mind that when you actually get to the region of the star from which photons escape, you do have to worry about more complicated physics (well – really – more precise treatment of the same basic physics).

For now, though, we can consider a box of thickness $dr$ with flux $H$ entering from the left, and $H - dH$ leaving on the right. Flux in this case means luminosity per surface area, so it is related to the flux $F$ we discussed earlier by the surface area – generally $4\pi r^2$ in stars.

Now, we can write down:

$$dH = -\kappa H \rho dr, \tag{16}$$

which defines $\kappa$, which is called the opacity coefficient – it's the opacity per unit mass (remember that since we are keeping track of density most of the time, because the gravitational force helps determine the stellar structure, we like to express things per unit mass rather than per atom).

We cna also write:

$$d\tau \equiv -\kappa\rho dr, \tag{17}$$

where $\tau$ is the optical depth. We define the optical depth to be zero at $\infty$, so it reaches a maximum at the center of the star.

Then, $H(r) = H_0 e^{-(\tau_0 - \tau(r))}$, where $\tau_0$ is the optical depth at the center of the star and $H_0$ is the flux at the center of the star. We can also then see that $(\kappa\rho)^{-1}$ is the mean free path length for a photon. Finally, we can use this to come up with a sensible definition for the radius of a star. As gas clouds, stars don't have fixed surfaces from which there is zero density, so instead, we define the radius to be the radius from which $\tau$ is unity. This is the radius from which the light "typically" comes.

*Interaction processes with light*

Let's remember that essentially all physical processes have an inverse. This gives us a useful way of figuring out what the absorption processes will be in a star – they're the opposite of key emission processes. There are six radiative absorption/scattering processes that are worth mentioning here. Two are common in stellar interiors, two are common in stellar atmospheres, and two are not common in stars, at all, but I'll mention them anyway. I'll list them in that order.

- Electron scattering. In the low energy regime, this is called Thomson scattering, and it changes the direction of a photon without changing its energy appreciably. In the high energy regime, it's called Compton scattering, and it also changes the energy, transferring some energy to the electron. Using the Compton scattering formulae for Thomson scattering gives the right answer, but it's a lot like using special relativity to do Newtonian mechanics problems – it is, strictly speaking, more correct, but the extra accuracy isn't worth the effort (or the computer time, if you're running a computer model of a star). Below $T \approx 10^{8-9}$K, Thomson scattering is a pretty good approximation of what's happening.

- Free-free absorption. This is the inverse process of bremsstrahlung radiation. When you have one charge interact with another, and accelerate it, it will emit a photon – that's bremsstrahlung – remember from electromagnetism that acceleration of charges (or magnetic fields) always produces radiation. In the presence of a photon field, the energies of some of the photons can be absorbed, and can increase the velocities of the particles, rather than decreasing them to emit light as bremstrahlung does.

- Bound-free transitions. This is another way of saying photoionization – the absorption of a photon to knock an electron out of an atom. This produces an "edge" in a spectrum – a feature which is sharp in one direction in wavelength/frequency space and falls off gradually. The reason is that the cross-section for interaction is maximum when the photon has exactly the right energy to make the ionization happen, but photons with extra energy can be absorbed (albeit with a lower probability) and then just leave the extra energy as kinetic energy for the electron.

- Bound-bound transitions. This is when a photon causes an electron to jump to an excited state, but one still in the atom. This causes a line in a spectrum.

- Pair production. This can be the interaction of two photons to produce an electron and positron, or a photon to scatter off an electron or positron and produce a pair. One needs 2 times the electron rest mass in the photons as a threshold for pair production to take place. These conditions are rare in stars, but when pair production happens in stars, the consequences can be profound – this can enable positron capture by nuclei, which can be quite important for the formation of certain chemical elements which are otherwise hard to form.

- Synchrotron/cyclotron radiation/absorption. This is the production or absorption of radiation through interactions with charged particles and magnetic fields. Cyclotron is the term used when the electrons are non-relativistic, so that the frequency of the radiation is essentially the frequency with which they orbit around magnetic field lines. Synchrotron radiation is the relativistic limit, where the higher harmonics of the orbit frequency are much more important. The magnetic fields inside stars are generally not large enough for synchrotron or cyclotron absorption to be important. In stellar coronae the magnetic fields can be large, and the particle energies can lead to strong cyclotron or synchrotron radiation.

Getting opacities is extremely tedious computational work. Usually grids are produced, and then scaling relations are developed. We'll usually talk about something called the "Rosseland mean" opacity for this course. Basically, this involves: (1) assuming that the photons and gas are at the same temperature, which can be shown[1] through careful numerical work to be an excellent assumption (2) putting in a blackbody spectrum, and getting the weighted mean opacity over the blackbody. Then, we do not need to calculate asborption coefficients as a function of wavelength. We can then just keep track of opacity as a function of density and temperature. It will turn out that power law approximations are pretty good for both.

Electron scattering is simple – it's just proportional to the Thomson cross section, which is independent of temperature and density. Then, $\kappa_{es} = \frac{1}{2}\kappa_{es,0}(1+X)$, with $\kappa_{es,0} = 0.04\ \mathrm{m}^2\ \mathrm{kg}^{-1}$. In the extreme relativistic regime, the cross-section for electron scattering drops a bit due to the "Klein-Nishina" correction, but that doesn't occur in normal stars.

For free-free absorption in a medium:

$$\kappa_{ff} = \frac{kappa_{ff}}{\mu_e}\left\langle\frac{Z^2}{A}\right\rangle\rho T^{-7/2} \tag{18}$$

with the same $\frac{1}{\mu_e} = \frac{1}{2}(1+X)$ substitution made here as for electron scattering. This is often called Kramers' opacity law, and we can substitute in $7\times10^{18}\mathrm{m}^5\mathrm{kg}^{-2}\mathrm{K}^{7/2}$.

I also then showed you a plot from the Los Alamos opacities tables for different values of $\rho$, and it was clear that there were some wiggles in the opacity due to atomic features (mostly where the ionization state of helium changes). For the most part, though, the curves are pretty smooth, and pretty well approximated by Kramers' law plus electron scattering.

[1]The phrase "can be shown" in books or journal articles or seminars almost always also means that it *won't* be shown, but could be, often in an extremely complicated mannner, and sometimes only by someone with more expertise than the presenter.

Now, we can check the numbers in a stellar interior and see how self-consistent some of our original assumptions are. E.g. inside the Sun, $\kappa$ is about 0.1 $\mathrm{m}^2/\mathrm{kg}$, and $\rho$ is about 1000 kg $\mathrm{m}^{-3}$, so the mean free path of a photon is about 1 centimeter. The temperature change over that range will turn out to be about $10^{-3}$K. This makes a blackbody a good approximation.

Now, let's consider what the temperature gradients really are. The rate at which photon flux is absorbed can be converted into a rate of momentum transfer – a force. We then do this per unit area and unit distance, and get a radiation pressure gradient:

$$\frac{H\kappa\rho}{c} = -\frac{dP_{rad}}{dr} \tag{19}$$

Next, we substitute in for the radiation pressure, and substitute $F = 4\pi r^2 H$, and get:

$$\frac{dT}{dr} = -\left(\frac{3}{4ac}\right)\left(\frac{\kappa\rho}{T^3}\right)\left(\frac{F}{4\pi r^2}\right) \tag{20}$$

One thing to note here is that if the radiation pressure gradient is too large, then it actually stops being possible to transfer energy that quickly through radiation alone, and particle motion is actually driven. This leads to convection in stellar interiors, and stellar winds in some extreme cases at the exteriors of stars. We'll come back to this later, but essentially when the temperature gradient is "too large" then we get convection.