Week 2: Basic equations of stellar structure

This week, we will go through the basic equations which govern the structure of stars, and try to develop some intuition about what we can learn from the stars themselves based on these equations and the physics behind them. I will follow closely Prialnik's chapter two, both in organization and in notation. I won't go through all the steps of the derivations in these notes – you can look them up in the book. I'll go over things which are necessary for following the basic physics, and some other equations where they're easy to typeset, but at some point, I'd just be copying and annotating the textbook for you. Also note that while I have done my best to avoid errors here, if you see a difference between what I do and what the book does, and I don't make a big deal of explaining it, assume the book is correct.

First, let's consider one of the most important concepts for understanding stars, that of local thermodynamic equilibrium. Two conditions must be met for a region of space to be in local thermodynamic equilibrium:

(1) the mean free path of the particles must be much less than the size scale of the system
(2) the mean free time of the particles must be much shorter than the time scale on which other properties of the system change

Deep in the interiors of stars, these conditions will almost always be met. In stellar coronae, for example, though, the conditions might not be met. There, the injection of energy from magnetic reconnection events can be highly variable, and the density of particles is quite low. An additional concern is that in some cases, there can be multiple equilbrium distributions, if e.g. electrons interact effectively with other electrons, but not with protons, or if charged particles interact with one another effectively, but not with the photons. Particles will follow the Maxwellian distribution when they are in thermal equilbrium, while photons will follow the Planck distribution if they are in equilbrium. It is fortunate that local thermodynamic equilbrium is usually such a good approximation of reality, since it is an extremely convenient approximation, allowing us to summarize distributions of large numbers of particles with a single quantity, the temperature.

Next, we can consider what else we need in order to describe the structure and evolution of a star. The evolution of a star can be described as a series of structures of a star, with one varying smoothly to the next. For the purposes of this course we will consider only the evolution of non-rotating (and hence spherical) stars, and will not consider the effects of magnetic fields. Rotation can affect the evolution of a stellar interior, but the effects are quite subtle, and extremely difficult to calculate, and so they don't make much sense to consider in a short course on stellar evolution – it's just good to be aware of this fact if you eventually go deeper in studies of stellar evolution.

So, given those simplifications, we basically need to solve a series of equations for the mass density as a function of radius and of time; the temperature as a function of radius and time; and the abundances of the different chemical elements as a function of radius and time. This gives $n + 2$ equations, where n is the number of chemical elements one follows in the solution. One should keep track of all the chemical elements up to the heaviest element that will be produced in a given star, and which element this is will depend primarily on the initial mass of the star. For very low mass stars, for example, only hydrogen and helium are relevant (even though other elements are present). Note that in some cases, it makes more sense to integrate over mass, rather than radius, and quantities can be expressed as functions of the mass enclosed, rather than as functions of the radius.

Next, one must consider what constraints exist on the structure of a star, and how they lead to equations and boundary conditions we can use to help solve the evolution of stars. We must conserve mass, energy, momentum, and angular momentum for a star in an equilbrium configuration. Angular momentum is conserved in a simple manner for the purposes of this course, by ignoring stellar rotation entirely, and treating the problem as spherically symmetric.

Mass conservation is also straightforward, and comes from:

$$dm = \rho dV = 4\pi r^2 \rho dr. \tag{1}$$

Implementation of the conservation of energy and of linear momentum will be considerably less trivial. Toward the end of the week's lectures, when we get to key timescales, we will simplify how we handle those as well, but it makes sense to understand a more formal, general approach to looking at these things, so that when we get to episodes in the life of a star where we leave the comfortable region of the main sequence, it will be nice not to have to re-learn everything that goes on in stars without simplifying assumptions made.

We will use δf to denote a change in the quantity f that occurs in a finite mass element over a small period of time δt. Then, we can look at conservation of energy:

$$\delta(udm) = dm\delta u = \delta Q + \delta W, \tag{2}$$

where u is the internal energy per unit mass, dm is the differential mass element's mass, Q is the heat absorbed, and W is the work done on a fluid element. We also use conservation of mass here, leaving dm constant.

We can then further define $F(m)$ to be the flux of heat flowing perpendicularly through a surface, and q to be the power per unit mass generated within dm. This has units of luminosity, and is not flux in the sense of flux observed from a telescope (which is luminosity per unit area). Working through some algebra (done in more detail in Prialnik), we find:

$$\dot{u} + P\dot{\left(\frac{1}{\rho}\right)} = q - \frac{\partial F}{\partial m}. \tag{3}$$

That is, the range of change in the internal energy plus the power applied to the element is equal to the power generated minus the gradient in the luminosity with respect to mass. In thermal equilibrium, the left hand side vanishes, and we get $q = \frac{dF}{dm}$ – the heat flow is just the energy generated locally. We can then integrate q over the whole star and define that to be the nuclear luminosity, L_{nuc}, and integrate $\frac{dF}{dm}$ over the whole star and get the total luminosity of the star, which are going to be equal to one another if the star is in thermal equilibrium. Main sequence stars are in thermal equilibrium, while some more evolved stars can deviate from thermal equilibrium, with gravity temporarily providing most of the power for the star as it changes its structure.

Equations of motion

Ok - now we're on to conservation of linear momentum – i.e. making sure that we have a stable structure to the star. In fact, this is simple, in some sense, since the star stays symmetric, so its net momentum will be zero, but let's also consider the one dimensional stability of the star, and make sure that it doesn't expand or contract too quickly. We can then write:

$$\ddot{r}\Delta m = -\frac{Gm\Delta m}{r^2} - \frac{\partial P}{\partial r}\frac{\Delta m}{\rho}, \tag{4}$$

where $\ddot{r}$ is the second time derivative of the radius of the star at a particular position, G is the gravitational constant, m is the mass of the star, Δm is the mass of a small piece of fluid in the star, r is the star's radius at a particular position, P is the pressure, and ρ is the density – essentially, the force is composed of the attractive gravitational force, and a term due to the pressure gradient, which will normally (though not strictly necessarily)[1] hold up the star, since the pressure will normally decrease outwards.

In hydrostatic equilibrium, the left side vanishes, and $\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$, or $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$. We can get the pressure at the center of the starby integrating this equation over the whole mass of the star, since the pressure at the outside of the star will be zero. If we do not know anything about the internal structure of the star, we can set r in the integrand to R, where R is the full radius of the star, and we can get a strict lower limit. For the Sun, this turns out to be 450 million atmospheres – and remember this is a strict lower limit.

Total energies

We can now apply some basic physics to look at stars' total energies and try to get a feel for certain properties they have. You've seen the virial theorem before several times, so I'll skip through a lot of what's in the book, just as I did in the lectures. Let's just take the key result:

$$U = -\frac{1}{2}\Omega, \tag{5}$$

where U is the internal energy of the system, and Ω is the gravitational binding energy of the system. Note that this formulation of the virial theorem is specific to gravitation and non-relativistic thermal particles.

[1] If the pressure increases as one moves outwards, the star is necessarily out of equilibrium. In some pulsations in otherwise stable stars, the pressure can increase outwards in the envelope of the star for parts of pulsational cycles, while the interior of the star remains in proper equilibrium.

Now, we can set $\Omega = -\alpha \frac{GM^2}{R}$, where α is a constant that depends on the density profile of the star. It will be $\frac{3}{5}$ for a constant density sphere.

We can then integrate U by noting that it's just $\frac{3}{2}k_BT$ times the number of particles, and can derive an expression for the mean temperature of the star, which turns out to be about 4 million Kelvin for a solar-like star. This is much higher than the surface temperature, and high enough that nearly all chemical elements will be fully ionized – it's thus useful for getting some idea of what messy things we'll need to worry about in the stellar interior, and which ones we probably won't have to worry about.

Next, we can go through another derivation, one which gives us the total energy of the star:

$$\int_0^M \dot{u} dm + \int_0^M P \dot{\left(\frac{1}{\rho}\right)} dm = L_{nuc} - L. \tag{6}$$

As a reminder, u is the internal energy per unit mass, P is the pressure and ρ is the density, with all quantities taken in a shell of differential mass unit dm with a radius such that mass m is enclosed.

We go through a series of rearrangements, explained well in the book, and tedious to re-type.

We end up with

$$\dot{U} - \int_0^M 4\pi r^2 \dot{r} \frac{\partial P}{\partial m} dm = L_{nuc} - L. \tag{7}$$

On the left side we have terms for the rate of change of the internal energy of the star and the rate at which work is done on the star, while on the right hand side, we have the nuclear luminosity minus the total luminosity.

Next, we can take the equation of motion for the star:

$$\int_o^M \dot{r}\ddot{r} dm = -\int_0^M \frac{Gm}{r^2} dm - \int_0^M 4\pi r^2 \dot{r} \frac{\partial P}{\partial m} dm. \tag{8}$$

Looking at this equation through dimensional analysis, we can see that the left hand side is giving us a velocity times acceleration on a differential mass element – force per unit mass times velocity integrated over the mass of the star – while the right hand side has a first term related to the change in the binding energy, and the last term is the change in the internal energy.

We can then define a new quantity, $\kappa = \int_0^M \frac{1}{2}\dot{r}^2 dm$, which is the bulk kinetic energy of the star, related to the whole radius of the star changing. We can then notice that the term related to the change in binding energy is $\dot{\Omega}$ and that the term related to the change in internal energy is $\dot{U}$. Equation 8 then can be re-written as:

$$\dot{U} + \dot{\kappa} + \dot{\Omega} = L_{nuc} - L, \tag{9}$$

or, we can see that $\dot{E}$, on the left is equal to $L_{nuc} - L$, or that the change in the total energy of the star is equal to the difference between the nuclear luminosity – the energy produced – minues the luminosity radiated, so that in equilibrium, we obtain a result we've already seen from other arguments, that the nuclear luminosity is equal to the total luminosity.

In hydrostatic equilibrium, we can also see that $\dot{Kappa}$ must be zero, since the star cannot be expanding or contracting quickly. That brings us back to the virial theorem, with the interesting point that $E = -U$ for stars. Thus is the star has negative heat capacity and if the internal energy increases, the star contracts. In the long run, this is the fate of all stars – the core contracts, the star heats up, and the envelope expands. The same thing happens for star clusters.[2]

Energy sources in stars

OK, now let's get on to where the energy in stars comes from, and what we can learn from this. We have been assuming all along in this course that the energy in stars comes from nuclear reactions, and on the main sequence, this is true. For massive stars, gravity can actually dominate over the lifetime of the star, because a tremendous amount of energy can be released from a supernovae explosion – often more than is radiated during the whole main sequence lifetime of the star.

Let's start with some basics of nuclear physics:

There are *baryons* and *leptons*. Baryons are heavy particles (protons and neutrons) while leptons are light particles (electrons, positrons, neutrinos and anti-neutrinos). There are other members of these classes of particles, but they're not important in stars, so we'll skip over them. We define particles in terms of mass, A and charge, Z, with A in atomic mass units and rounded to the nearest integer and Z in units of the fundamental charge.

Nuclear reactions must conserve the usual suspects – mass, energy, charge, angular momentum, and momentum, but also must conserve baryon number and lepton number. The main consequence of the lepton number conservation is the production of neutrinos and/or anti-neutrinos when neutrons and protons are converted into one another. Basically, the neutrino or anti-neutrino cancels out the effects of producing an electron or positron in this reaction. We don't have to conserve photon number, so if nuclear binding energy is released as photons (usually gamma-rays), that's fine.

Next, we can try to figure out how to put nuclear reactions into a stellar model. We can define X_i as the mass fraction of species i, where i is a particular isotope of a particular element. We can write $X_i = \frac{\rho_i}{\rho}$.

Note that masses of most atomic nuclei are a little smaller than the masses of A protons, but we pretend that they aren't when we're balancing nuclear reactions. Obviously when computing the energy produced, we do need to sort out the right masses. Strangely, we use m_H as the notation for the atomic mass unit, implying that it's the proton mass, but we actually use $\frac{1}{12}$ of the mass of the ^{12}C nucleus.

[2]An interesting point, which may not get to in class, is that globular clusters must have an extra energy source, too. This turns out to be the binding energy locked up in binary stars, which can be released in close encounters. Otherwise all globular clusters would be "core-collapsed" by now.

We can then write $n_i = \frac{\rho X_i}{m_H A_i}$, where n_i is the number density in species i, ρ is the total mass density, m_H is the amu, A_i is the baryon number for species i and X_i is the mass fraction in species i.

In writing out a nuclear reaction from $I(A_i, Z_i) + J(A_j, Z_j)$ to $K(A_k, z_k) + L(A_l, z_l)$, we need to keep $A_i + A_j = A_k + A_l$ and $Z_i + Z_j = Z_k + Z_l$, although we have some wiggle room in the charge if we change neutrons and protons into one another, releasing electrons or positrons and neutrinos or anti-neutrinos.

Now, we need to look at the rate at which nuclear reactions occur. This will depend on a lot of things – the four most important are the gas density, the gas temperature, the energy barrier for starting the reaction, and the energy released by having the reaction take place. We need to consider, in principle, the forward reaction and the backward reaction, although it is rare that the rates of these are remotely comparable.

The number of nuclear reactions per unit time will be:

$$n_i n_j \sigma v = n_i n_j R_{ijk}, \tag{10}$$

where σ is the cross section for the forward reaction, and v is the thermal velocity of the particles. σv is defined to be R_{ijk}, the reaction rate coefficient. Note that in practice, σ will often be a function of v, so this approach is a simplififcation. Note that Prialnik uses an alternative representation of σ in the book, the form that is used at the end of a word in Greek. In my notes, I have changed this.

The total evolution of a species can be obtained by looking at the rates of the forward and backward reactions. Note that you need to be careful not to double-count particles in cases where a species is reacting with a like particle.

Evolution equations

So, now look at the key evolution equations for stars. We'll then look at how we can simplify them by deriving some key timescales for the evolution of stars that allow us to "cheat" our way into throwing away certain terms.

(1) basically something like momentum conservation, albeit by turning an inherently 3D equation into a 1D equation:

$$\ddot{r} = -\frac{Gm}{r^2} - 4\pi \frac{\partial P}{\partial m} \tag{11}$$

This says that the acceleration of the size scale of the star comes from comparing gravity inwards and pressure gradients acting outwards.

(2) This is basically energy conservation.

$$\dot{u} + P(\dot{1\rho}) = q - \frac{\partial F}{\partial m} \tag{12}$$

(3) We can "vectorize" the nuclear reaction equations. These are actually the simplest ones to deal with, since these all come from basic physics (although there are still cross sections relevant for stars but which aren't well measured in reactors):

$$\dot{\boldsymbol{X}} = \boldsymbol{f}(\rho, T, \boldsymbol{X}) \tag{13}$$

where $\boldsymbol{f}$ contains all the rates of change of all the different elements, enclosed in the elements of $\boldsymbol{X}$. Another way to look at this equation is that it contains an awful lot of stuff to compute, and some of the numbers aren't well known, but all the basic physics of what should happen in here is just nuclear physics rate constants – we don't need any additional assumptions about how stars behave for this one.

OK - so this is a framework for solving stellar structure and evolution, but we have a lot more equations than constraints, so we either need some tricks to add more constraints, or we need initial conditions for how the star starts to form.

Looking at our equations, we have two space derivatives, and $n+3$ time derivatives, where n is the number of isotopes we're tracking. The boundary conditions for the space derivatives are really simple – there's no pressure at the outer radius of the star, and there's no energy flux at the center of the star. The nuclear reactions are well settled, in principle, if not in practice[3]. We still have three time derivatives, though, that aren't "easy" to deal with, but looking at some key timescales will show us the clever tricks we can use to dodge the problem.

Characteristic timescales

A characteristic timescale can be defined as $\tau = \frac{\phi}{\dot{\phi}}$, where ϕ is some quantity of interest.

The dynamical timescale is the timescale on which the radius of the star changes, so for this one, $\phi = R$. Roughly speaking, the time derivative will be the freefall velocity – the rate at which the star would collapse in the absence of pressure. $\tau_{dyn} = \frac{1}{\sqrt{G\rho}}$ gives this up to small constants. It's about 1000 seconds for the Sun.

This timescale is normally the timescale on which perturbations to the star sort themselves out. It's also interesting for understanding the fastest global oscillations of a star, and for understanding just how fast supernovae take place.

The thermal timescale is the timescale on which the star would use up its heat reservoir if the fuel source were turned off. Here, $U = \phi$. The energy reservoir can be taken as $\frac{GM^2}{R}$, and the rate of its change is the luminosity, L. So, $tau_{th} = \frac{GM^2}{RL}$. This is of order 30 Myrs for the Sun. The relation:

$$tau_{th} = 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{M_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} \tag{14}$$

given in the textbook comes from using the main sequence mass-luminosity relation, $L \propto M^4$.

[3]One of the key areas of research in experimental nuclear physics in the past decade or two has been measuring the cross-sections of reactions that are important for astrophysics

This timescale is short compared to the stellar lifetime, which means that as the star's luminosity changes and its temperature distribution changes, the star has time to adjust its structure.

The nuclear timescale is the slowest of the key timescales. It comes from taking the total energy reservoir for the star and dividing by the luminosity. We take:

$$\tau_{nuc} = \frac{\eta M c^2}{L} \tag{15}$$

where η is the fraction of the rest mass that is converted to energy in nuclear reactions, and the other symbols have their usual meanings.

This number can obviously scale strongly with stellar mass, but it's always much larger than the thermal timescale.

Now, the fact that the nuclear timescale is so much longer than the other timescales means that we can throw away a bunch of the other terms when computing a stellar model, because all that matters is the nuclear timescale. The whole structure of the star is determine by the nuclear reactions in the center.

The equation describing the nuclear reactions was already acceptable, and is unchanged. The first evolution equation can be simplified by setting $\ddot{r}$ to zero, so that the pressure gradient is just given by balancing the gravitational force. The second equation can be simplified by setting the rate of energy change to zero, so that the flux gradient is just the energy generated locally.

Another important thing to think about is that what this all means is that when only τ_{nuc} matters, then the star's equilibrium configuration depends only on the current mass and composition – and not how the star was assembled. This is good, for two reasons. One is that star formation is still not terribly well understood, and the other is that we know we have the main sequence, so it's reassuring that this is a consequence of the basic equations that govern stellar structure and evolution and not some conspiracy in terms of all stars forming in exactly the same way.