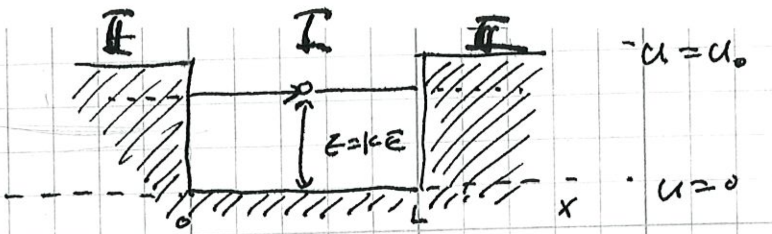


② Region I $(0 < x < L)$

since $U(x) = 0$ here,

the time-independent Schrödinger eq

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \cancel{U(x)}\psi(x) = E\psi(x)$$

$$\rightarrow \frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}E\psi(x)$$

for convenience, let's make the following definition.

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}}$$

$$\rightarrow \frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$$

$\frac{4m \cdot \frac{1}{4}mv^2}{\hbar^2} = \frac{m^2v^2}{\hbar^2}$ $\left(\frac{mv}{\hbar}\right)^2$

$p = \hbar k$ $\frac{p}{\hbar} = \sqrt{k}$

$k = \frac{p}{\hbar}$

Solution: $\psi(x) = A\sin kx$ & $\psi(x) = B\cos kx$

$$\Rightarrow \psi(x) = A\sin kx + B\cos kx$$

General solution for region I

⑥ Region II $(x < 0)$

Here, $U(x) = U_0$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U_0\psi(x) = E\psi(x)$$

$$\rightarrow \frac{d^2\psi(x)}{dx^2} = \frac{2m(U_0 - E)}{\hbar^2}\psi(x), \quad \alpha \equiv \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

$$\rightarrow \frac{d^2\psi(x)}{dx^2} = \alpha^2\psi(x)$$

positive

A pair of independent solution is,

$$\psi(x) = Ce^{+\alpha x}, \quad \psi(x) = De^{-\alpha x}$$

and the general solution is the sum

$$\psi(x) = Ce^{+\alpha x} + De^{-\alpha x}$$

diverges as $x \to -\infty$

i.e. mathematically o but physically unacceptable

so $D = 0$

○ Region III $(x > L)$

Same as Region I (since $U(x)$ is the same)

$$\psi(x) = F e^{+\alpha x} + G e^{-\alpha x}$$

↳ F must be "zero"

$x \to \pm\infty$

$\psi \to \infty$

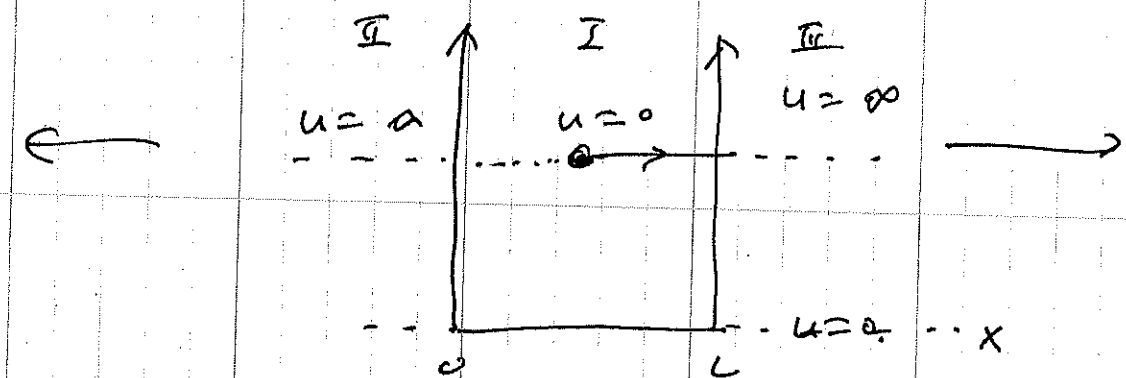
$\Rightarrow F = D = 0$

$$\boxed{\psi(x) = \begin{cases} C e^{+\alpha x} & x < 0 \\ A \sin kx + B \cos kx & 0 < x < L \\ G e^{-\alpha x} & x > L \end{cases}}$$

(1.)

where $k \equiv \sqrt{\dfrac{2mE}{\hbar^2}}$, $\alpha \equiv \sqrt{\dfrac{2m(U_0 - E)}{\hbar^2}}$

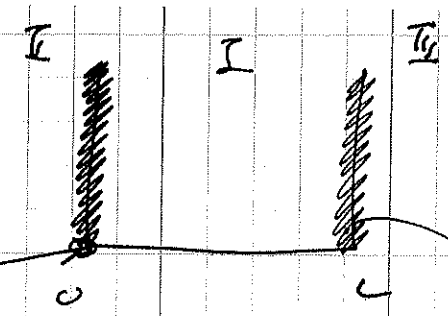
** ptl in a Box: The Infinite Well



$$\psi(x) = \begin{cases} A \sin kx + B \cos kx & 0 < x < L \\ 0 & x < 0,\ x > L \end{cases}$$

$A = ?$

$B = ?$



If wave f-t is to be continuous,
its pieces must have the same value @ $x=0$,

$$\psi_{II}(0) = \psi_{I}(0)$$

or

using $\psi(x) = A\sin kx + B\cos kx$

$$\Rightarrow \phi = \underbrace{A\sin k\phi}_{0} + \underset{\substack{\uparrow \\ \text{must be } 0}}{B}\underbrace{\cos k\phi}_{1} \rightarrow B = 0$$

see

now $x = L$,

$$\psi_I(L) = \psi_{III}(L)$$

or

$$\psi(x) = A\sin kx + B\cos kx$$

B is already zero here
now forget about B

$$0 = A\sin kL$$

The only the wave f-t can be continuous is if the following condition hold:

$$\boxed{kL = n\pi}$$

$$\sqrt{\frac{2mE}{\hbar^2}}\,L = n\pi \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$\underline{\psi(x) = A \sin kx}$

$$\boxed{\begin{aligned} \therefore \quad \psi_n(x) &= A \sin \frac{n\pi}{L} x \qquad (0 < x < L) \\ E_n &= \frac{n^2 \pi^2 \hbar^2}{2mL^2} \end{aligned}}$$

$\leftarrow$ only certain E-levels are allowed!!

☆☆ The wave f.t is of typical standing-wave form!!
QMcally, bound state are standing wave!!

Thus, we have the normalized, continous W.f representing the allowed states of ptl in an infinite well and their corresponding Es!!

$$\boxed{\begin{aligned} \psi_n(x) &= \begin{cases} \sqrt{\dfrac{2}{L}} \sin \dfrac{n\pi x}{L} & 0 < x < L \\ 0 & x < 0, \quad x > L \end{cases} \\ E_n &= \frac{n^2 \pi^2 \hbar^2}{2mL^2} \end{aligned}}$$

A we will see this soon

$A = ?$

$\int_0^L \sin^2\left[\left(\frac{n\pi}{L}\right)x\right] dx = \frac{L}{2}$

☆ Normalization of $\psi(x,t)$

$$\int_{\text{all space}} |\psi(x,t)|^2 dx = 1$$

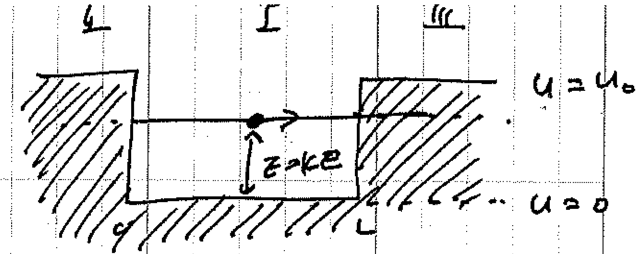
see Appendix K-5

$$\frac{x}{2} - \frac{L}{4n\pi} \sin\left(\frac{2n\pi x}{L}\right)\Big|_0^L$$

$$\int_{\text{all space}} |\psi(x,t)|^2 dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_0^L \left(A \sin \frac{n\pi}{L} x\right)^2 dx = 1$$

$$\therefore \quad A^2 \frac{L}{2} = 1 \quad \rightarrow \quad A = \sqrt{\frac{2}{L}}$$

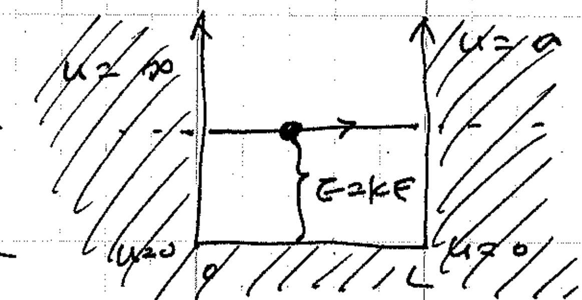
Finite Potential Well



$$\psi(x) = \begin{cases} \psi_{II}(x) & Ce^{+\alpha x} & x < 0 \\ \psi_I(x) & A\sin kx + B\cos kx & 0 < x < L \\ \psi_{III}(x) & Ge^{-\alpha x} & x > L \end{cases}$$

where $k \equiv \sqrt{\frac{2mE}{\hbar^2}}$, $\alpha \equiv \sqrt{\frac{2m(u_0 - E)}{\hbar^2}}$

Infinite Potential Well



$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} & 0 < x < L \\ 0 & x < 0, \; x > L \end{cases}$$

$$E_n(x) = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

DONE !!

Back to Finite Potential Well problem.

A, B, C, G, E ? — energy

Let's look at "Continuity" at $x = 0$, $= L$

$\Rightarrow$ $\psi(x)$ continuous @ $x = 0$: $\psi_{II}(0) = \psi_I(0)$

$\therefore$ $C\underbrace{e^{+\alpha 0}}_{1} = A\underbrace{\sin k0}_{0} + B\underbrace{\cos k0}_{1} \Rightarrow \boxed{C = B}$

$\Rightarrow$ $\frac{\partial \psi(x)}{\partial x}$ continous @ $x = 0$: $\left.\frac{\partial \psi_{II}}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_I}{\partial x}\right|_{x=0}$

$\therefore$ $\alpha C\underbrace{e^{+\alpha 0}}_{1} = kA\underbrace{\cos k0}_{1} - kB\underbrace{\sin k0}_{0} \Rightarrow \boxed{\alpha C = kA}$

$\rightarrow \psi(x)$ continous @ $x = L$: $\psi_{I}(L) = \psi_{III}(L)$

$$A \sin kL + B \cos kL = G e^{-\alpha L}$$

$\rightarrow \frac{\partial \psi(x)}{\partial x}$ continous @ $x = L$: $\left.\frac{\partial \psi_I}{\partial x}\right|_{x=L} = \left.\frac{\partial \psi_{III}}{\partial x}\right|_{x=L}$

$$kA \cos kL - kB \sin kL = -\alpha G e^{-\alpha L}$$

we know two things : $\boxed{\begin{array}{l} C = B \\ \alpha C = kA \end{array}} \rightarrow A = \frac{\alpha C}{k}$

$$\frac{\alpha}{k} C \sin kL + C \cos kL = G e^{-\alpha L} \quad \text{—— ①}$$

$$\left(k \frac{\alpha C}{k} \cos kL - kC \sin kL\right) / (-\alpha) = (-\alpha) G e^{-\alpha L} / (-\alpha) \quad \text{—— ②}$$

To make equal b/w ①-②

$$\frac{\alpha}{k} \cancel{C} \sin kL + \cancel{C} \cos kL = -\cancel{C} \cos kL + \frac{k}{\alpha} \cancel{C} \sin kL$$

$\Downarrow$

$$\frac{\alpha}{k} \sin kL + \cos kL = -\cos kL + \frac{k}{\alpha} \sin kL$$

$$\left(\frac{k}{\alpha} - \frac{\alpha}{k}\right) \sin kL = \cos kL + \cos kL \; (= 2\cos kL)$$

$$\left(\frac{k}{\alpha} - \frac{\alpha}{k}\right) = 2\left(\frac{\cos kL}{\sin kL}\right) = 2 \cot kL$$

$$\boxed{2 \cot kL = \frac{k}{\alpha} - \frac{\alpha}{k}}$$

Transcendental Eq.

impossible to solve analytically !!

But numerical solution is possible

$\Rightarrow$ e.g. graphing solution !!

$$2\cot kL = \frac{k}{\alpha} - \frac{\alpha}{k}$$

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

~~sec~~ / secant

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

cosecant

$$2\cot kL = 2\cdot\frac{\cos kL}{\sin kL} = 2\cdot\frac{\cos^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{kL}{2}\right)}{2\sin\left(\frac{kL}{2}\right)\cos\left(\frac{kL}{2}\right)}$$

$$\left(\cot\left(\frac{kl}{2}\right) - \tan\left(\frac{kl}{2}\right)\right) = \frac{k}{\alpha} - \frac{\alpha}{k}$$

Multiplying by αk

$$\alpha k\left(\cot\left(\frac{kl}{2}\right) - \tan\left(\frac{kl}{2}\right)\right) = \alpha k\left(\frac{k}{\alpha} - \frac{\alpha}{k}\right)$$

$$= k^2 - \alpha^2$$

$$\alpha^2 + \alpha k\left(\cot\left(\frac{kl}{2}\right) - \tan\left(\frac{kl}{2}\right)\right) - k^2 = 0$$

Factoring, $\left(\alpha + k\cot\left(\frac{kL}{2}\right)\right)\left(\alpha - k\tan\left(\frac{kL}{2}\right)\right) = 0$

$$\rightarrow \boxed{\alpha = -k\cot\left(\frac{kL}{2}\right) \text{ or } k\tan\left(\frac{kL}{2}\right)}$$

odd

$$\alpha \equiv \frac{\sqrt{2m(U_0 - E)}}{\hbar}, \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\alpha = -k\cot\left(\frac{kL}{2}\right)$$

$$\frac{\sqrt{2m(U_0-E)}}{\hbar} = -\frac{\sqrt{2mE}}{\hbar}\cot\left(\frac{kL}{2}\right)$$

$$U_0 - E = E\cot^2\left(\frac{kL}{2}\right)$$

$$U_0 = E\left(\cot^2\left(\frac{kL}{2}\right) + 1\right) = E\csc^2\left(\frac{kL}{2}\right)$$

$$\alpha = +k\tan\left(\frac{kL}{2}\right) \Rightarrow \frac{\sqrt{2m(U_0-E)}}{\hbar} = \frac{\sqrt{2mE}}{\hbar}\cdot\tan\left(\frac{kL}{2}\right)$$

$$U_0 - E = E\tan^2\left(\frac{kL}{2}\right)$$

$$U_0 = E\left(\tan^2\left(\frac{kL}{2}\right) + 1\right) = E\sec^2\left(\frac{kL}{2}\right)$$

Replacing E w/ $\frac{\hbar^2 k^2}{2m}$ $\longrightarrow$ become (5-23)

$$U_0 = \begin{cases} \frac{\hbar^2 k^2}{2m}\sec^2\frac{kL}{2} & (n-1)\pi < kL < n\pi, \quad n = 1, 3, 5 \dots \\ \frac{\hbar^2 k^2}{2m}\csc^2\frac{kL}{2} & (n-1)\pi < kL < n\pi, \quad n = 2, 4, 6 \dots \end{cases}$$

* Graphing Solution

$$2\cot kL = \frac{k}{\alpha} - \frac{\alpha}{k}$$

$$\alpha = \begin{cases} k\tan\frac{1}{2}kL & 2n\pi < kL < 2n\pi + \pi \\ -k\cot\frac{1}{2}kL & 2n\pi - \pi < kL < 2n\pi \end{cases}$$

$$\sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

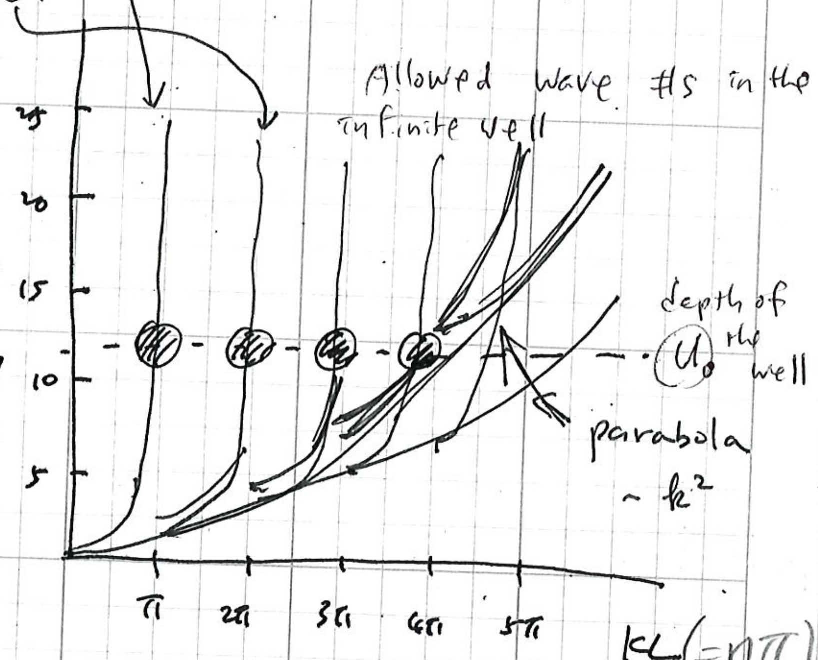
$$U_0 = \begin{cases} \frac{\hbar^2 k^2}{2m}\sec^2\frac{1}{2}kL, & \underset{(n-1)\pi}{2n\pi} < kL < \underset{n\pi}{2n\pi+\pi} \quad n=1,3,5\ldots \\ \frac{\hbar^2 k^2}{2m}\csc^2\frac{1}{2}kL & \underset{(n-1)\pi}{2n\pi-\pi} < kL < \underset{n\pi}{2n\pi} \quad n=2,4,6\ldots \end{cases}$$

$U_0 = E\sec^2\frac{kL}{2}$

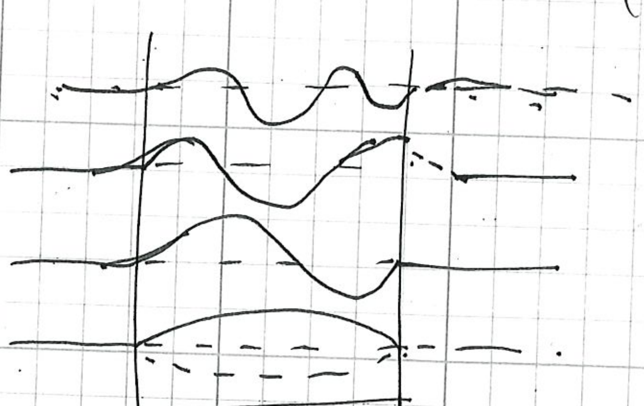
$E = \frac{U_0}{\sec^2\left(\frac{kL}{2}\right)}$

$E =$ 5 0,

$E_n \left(\frac{\pi^2\hbar^2}{2mL^2}\right)$



Finite well			Infinite well	
$U_0 = 12\frac{\pi^2\hbar^2}{2mL^2}$			$U_0 = \infty$	
n	k	E	k	E
1	↓	↓	↓	↓
2				
3				
4				



$E = \frac{\hbar k^2}{2m}$,

$E = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$\boxed{kL = n\pi}$

$= 0, \pi, 2\pi, 3\pi, \ldots$

Volunteer work

$$2\cot kl = \frac{k}{\alpha} - \frac{\alpha}{k}, \qquad k \equiv \sqrt{\frac{2mE}{\hbar^2}}, \qquad \alpha \equiv \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

limit $U_0 \to \infty \Rightarrow \alpha \to \infty$

then $2\cot kl = \frac{k}{\alpha} - \frac{\alpha}{k}$ becomes $\cot kl = -\infty$

this requires

$$\boxed{kl = n\pi}$$

Same as infinite well condition.

$\Rightarrow$ the infinite well is indeed a limiting case of the finite well. ($U_0 \to \infty$)