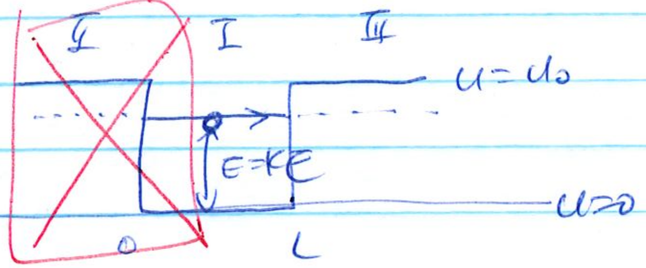
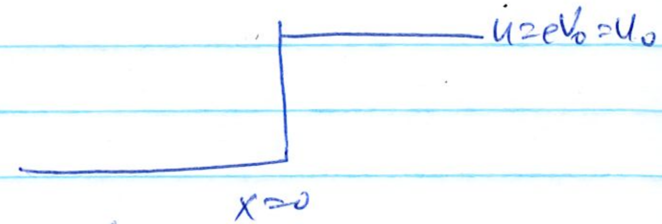
Ch6. "Un" bound states

"Potential step"



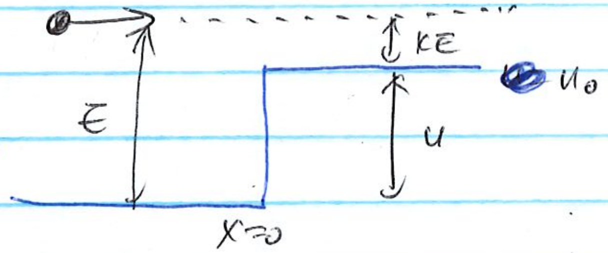


A Free electron encounters a potential step!!

Two possible cases

Case I

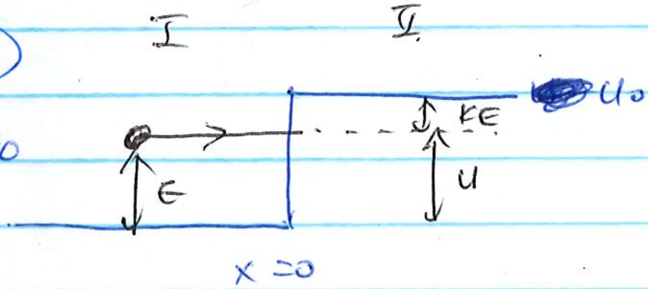
$E > U_0$



$$U(x) = \begin{cases} 0 & x<0 \\ U_0 & x>0 \end{cases}$$

Case II

$E < U_0$



from potential well

$$\boxed{\frac{d^2\psi(x)}{dx^2} = \left[\frac{2m(U(x)-E)}{\hbar^2}\right]\psi(x)}$$

$\boxed{1}$ Case I, first.

(1) Region I. $(x<0)$

negative $\because E > U_0$

$$\frac{d^2\psi(x)}{dx^2} = \left[\frac{2m(\cancel{U(x)}-E)}{\hbar^2}\right]\psi(x)$$

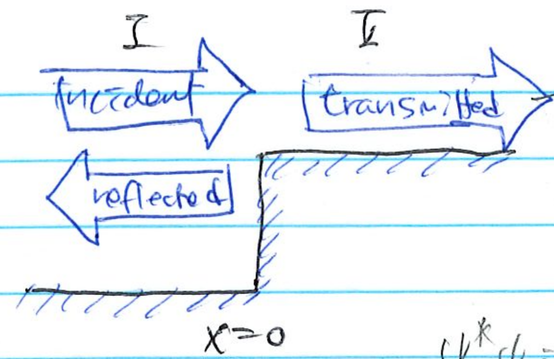
0

$$\Rightarrow \psi_I(x) = Ae^{+ikx} + Be^{-ikx}, \quad \text{where } k \equiv \sqrt{\frac{2mE}{\hbar^2}}$$

Combinations of two W.f

Incident ↙ ↘ reflected

What are these?



we'll see this later!!

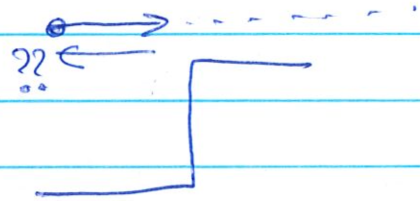
$$\psi^*\psi = A^* e^{-ikx} A e^{+ikx} = A^*A$$

Probability Density

inc $|\psi|^2_{inc} = A^*A$ $\qquad$ $|\psi|^2_{refl} = B^*B$ ref

~ proportional to the # of ptls per unit distance
on the left of the step
and moving to the right.

BTW, reflected ~~these~~ Is Classically possible?



↓

NO!! $\quad$ so, $|\psi|^2_{refl} = B^*B = 0$ in C.M.

(ii) Region II $(x>0)$.

i.e. $\dfrac{d^2\psi(x)}{dx^2} = \underbrace{\left[\dfrac{2m(U(x)-E)}{\hbar^2}\right]}_{<0 \ (\because E>U)}\psi(x)$

$\psi_{II}(x) = Ce^{+ik'x} + De^{-ik'x}$, where $k' \equiv \sqrt{\dfrac{2m(E-U_0)}{\hbar^2}}$ negative

beyond $x=0$, there is no change in P.E.
i.e. no force to cause a wave moving to right to reflect.

Because there can be no left-moving wave

$$\psi_{II}(x) = Ce^{+ik'x} + \cancel{De^{-ik'x}}$$

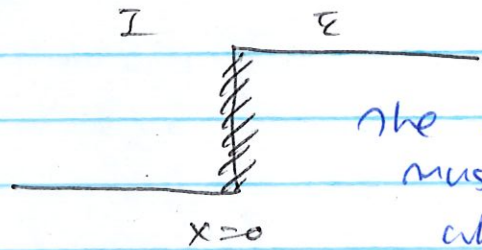
$$\Rightarrow \underbrace{\psi_{II}(x) = Ce^{+ik'x}}_{\text{transmitted } \psi} \quad , \text{ where } k' \equiv \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

☆☆ Probability density of finding a right-moving ptl is

$$|\psi|^2_{trans} = C^*C$$

transmitted

Let's find out A, B, C !!



The wave function and its derivates must be continuous @ the point where the two regions meet.

$\Rightarrow$ i.e. smoothness (continuity)
$$\begin{cases} \psi_I(0) = \psi_{II}(0) \\ \left.\frac{\partial \psi_I}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_{II}}{\partial x}\right|_{x=0} \end{cases}$$

$$Ae^{+ik0} + Be^{-ik0} = Ce^{+ik'0}$$

$$\Rightarrow \boxed{A + B = C}$$

$$ikAe^{+ik0} - ikBe^{-ik0} = ik'Ce^{+ik'0}$$

$$\Rightarrow \boxed{k(A-B) = k'C}$$

$$* \quad C = \left(\frac{2k}{k+k'}\right)B$$

$$B = \left(\frac{k-k'}{k+k'}\right)A$$

OK, shall we now talk about $\left\{\begin{matrix}\text{Transmission}\\ \text{Reflection}\end{matrix}\right\}$ Probability?

(i) Transmittion prob.

$$T = \frac{\frac{\#\ \text{transmitted}}{\text{time}}}{\frac{\#\ \text{incident}}{\text{time}}}$$

(ii) Reflection prob.

$$R = \frac{\frac{\#\ \text{reflected}}{\text{time}}}{\frac{\#\ \text{incident}}{\text{time}}}$$

$\frac{\#}{\text{time}}$ is proportional to what? $\propto$?

we know that $|\psi|^2 \propto$ (# of particles per unit distance)

$$\frac{\#}{\text{time}} = \underbrace{\boxed{\frac{\#}{\text{distance}}}}_{|\psi|^2} \ \underbrace{\frac{\text{distance}}{\text{time}}}_{v} \propto |\psi|^2 v$$

also we now that

$$v = \frac{p}{m} = \frac{\hbar k}{m} \Rightarrow v \sim k$$

(i)'

so, $$T = \frac{|\psi|^2_{trans}\, k_{II}}{|\psi|^2_{inc}\, k_I} = \frac{C^*C}{A^*A}\frac{k'}{k}$$

(ii)'

$$R = \frac{|\psi|^2_{refl}\, \cancel{k_I}}{|\psi|^2_{inc}\, \cancel{k_I}} = \frac{B^*B}{A^*A}$$

(i)"

$$T = \frac{C^* C}{A^* A}\ \frac{k'}{k}$$

$$= \frac{\left(\frac{2k}{k+k'}A\right)^* \left(\frac{2k}{k+k'}A\right)}{A^* A}\ \frac{k'}{k}$$

$$= \frac{4kk'}{(k+k')^2}$$

(ii)"

$$R = \frac{B^* B}{A^* A}$$

$$= \frac{\left(\frac{k-k'}{k+k'}A\right)^* \left(\frac{k-k'}{k+k'}A\right)}{A^* A}$$

$$= \frac{(k-k')^2}{(k+k')^2}$$

OK, $\boxed{T + R = ?}$ guess what !! — yes, shoud be $\underline{\underline{1}}$

let's check it out !!

$$T + R = \frac{4kk'}{(k+k')^2} + \frac{(k-k')^2}{(k+k')^2}$$

$$= \frac{4kk' + k^2 + k'^2 - 2kk'}{(k+k')^2} = \frac{(k+k')^2}{(k+k')^2} = \boxed{1}$$

//.

Bingo"

Finally, we express the prob.
in terms of U_0 & E
using the definitions of k, k'.

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}}$$

$$k' \equiv \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$$

(i)'''
$$T = 4\frac{\sqrt{E(E-U_0)}}{(\sqrt{E}+\sqrt{E-U_0})^2}$$

(ii)'''
$$R = \frac{(\sqrt{E}-\sqrt{E-U_0})^2}{(\sqrt{E}+\sqrt{E-U_0})^2}$$

$\boxed{2}$ Case II $(E<U_0)$