Summary

Time-independent Schrödinger equation

\[-\frac{\hbar^2}{2m} \nabla^2 \psi(r) + U(r)\psi(r) = E\psi(r)\]

Hydrogen atom potential energy

\[U(r) = -\frac{1}{4\pi \epsilon_0} \frac{e^2}{r}\]  \hspace{1cm} (6-9)

Hydrogen atom wave function

\[\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)\]

\[E = -\frac{me^4}{2(4\pi \epsilon_0)^2 \hbar^2 n^2} \quad (n = 1, 2, 3, \ldots)\]

\[|L| = \sqrt{\ell(\ell + 1)} \hbar \quad (\ell = 0, 1, 2, \ldots, n - 1)\]

\[L_z = m_l \hbar \quad (m_l = 0, \pm 1, \pm 2, \ldots)\]

Quantized so far:

The projection of angular momentum to z-axis

\[L_z = m_l \hbar \quad (m_l = 0, \pm 1, \pm 2, \ldots)\]

The magnitude of (orbital) angular momentum

\[|L| = \sqrt{L^2} = \sqrt{\ell(\ell + 1)} \hbar \quad (\ell = 0, 1, 2, \ldots)\]

The radial part of Schrödinger equation for hydrogen atom

\[\frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \frac{\cos^2 \theta}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{2m(E - U(r))}{\hbar^2}\]

\[\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \ell(\ell + 1) = -r^2 \frac{2m(E - U(r))}{\hbar^2}\]
The radial equation is called the associated Laguerre equation and the solutions \( R \) that satisfy the appropriate boundary conditions are called associated Laguerre functions.

Energy quantization

The principal quantum number \( n \)

\[
E = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2 n^2} \quad \text{(where } n = 1, 2, 3, \ldots \text{)}
\]

\( \ell = 0, 1, 2, \ldots, n - 1 \) and \( \ell \) must obey

Quantum number \( l \) is limited to \((n-1)\)

Bohr Radius

\[
a_0 = \frac{(4\pi\varepsilon_0)\hbar^2}{me^2} = 0.0529 \text{ nm}
\]

How Small is “Small”?
Quantum numbers

\[ \ell = 0, 1, 2, \ldots, n - 1 \]
\[ m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell \]

Because of \( 1/r \)

Energy

\[ n \]
\[ \ell \]
\[ m_\ell \]

\[ \begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
2 & 0 & -1 & 0 & +1 & 0 & -1 & \ldots & 0 & \ldots & n - 1 \\
3 & 0 & -1 & 0 & +1 & 0 & -1 & \ldots & 0 & \ldots & n - 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
n & 0 & -1 & 0 & +1 & 0 & -1 & \ldots & 0 & \ldots & n - 1 \\
\end{array} \]

Degeneracy

\[ \begin{array}{cccccc}
1 & 4 & 9 & \ldots & n^2 \\
\end{array} \]

Probability Density and Normalization

\[ |\Psi(r, \theta, \phi)|^2 = |\psi(r, \theta, \phi)|^2 = (R(r)\Theta(\theta)\Phi(\phi(t)))^2 (R(r)\Theta(\theta)\Phi(\phi(t))) \]

Total Prob. of finding electron somewhere in space must be “1”. Using the volume element in spherical polar coordinate (see 7-4), normalization condition becomes...

Hydrogen Atom Radial Wave Functions

<table>
<thead>
<tr>
<th>( n, \ell )</th>
<th>( R_{n, \ell}(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0</td>
<td>( \frac{1}{a_0} e^{-r/a_0} )</td>
</tr>
<tr>
<td>2, 0</td>
<td>( \frac{1}{2a_0} \left( 1 - \frac{1}{2a_0} \right) e^{-r/2a_0} )</td>
</tr>
<tr>
<td>2, 1</td>
<td>( \frac{1}{2a_0} r e^{-r/2a_0} )</td>
</tr>
<tr>
<td>3, 0</td>
<td>( \frac{1}{3a_0} \left( 1 - \frac{2}{3a_0} \right) e^{-r/3a_0} )</td>
</tr>
<tr>
<td>3, 1</td>
<td>( \frac{1}{6a_0} \left( 1 - \frac{3}{3a_0} \right) e^{-r/3a_0} )</td>
</tr>
</tbody>
</table>

Associated Leguerre Equation: \( R(r) = A e^{-r/a_0} \), \( \frac{d^2R}{dr^2} + \frac{2}{r} \frac{dR}{dr} - \ell(\ell + 1) = \frac{2m(E - U(r))}{\hbar^2} \)

Table 6.6 Angular Solutions (Spherical Harmonics)

<table>
<thead>
<tr>
<th>( \ell, m_\ell )</th>
<th>( \Theta_{\ell, m_\ell}(\theta) \Phi_{\ell, m_\ell}(\phi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 0</td>
<td>( \sqrt{\frac{2}{\pi}} )</td>
</tr>
<tr>
<td>1, 0</td>
<td>( \sqrt{\frac{3}{2\pi}} \cos \theta )</td>
</tr>
<tr>
<td>1, 1</td>
<td>( \sqrt{\frac{3}{2\pi}} \sin \theta \cos \phi )</td>
</tr>
<tr>
<td>2, 0</td>
<td>( \sqrt{\frac{5}{2\pi}} (3 \cos^2 \theta - 1) )</td>
</tr>
<tr>
<td>2, 1</td>
<td>( \sqrt{\frac{5}{2\pi}} \cos \theta \sin \phi )</td>
</tr>
<tr>
<td>2, 2</td>
<td>( \sqrt{\frac{15}{2\pi}} \sin^2 \theta \cos^2 \phi )</td>
</tr>
<tr>
<td>3, 0</td>
<td>( \sqrt{\frac{7}{2\pi}} (5 \cos^3 \theta - 3 \cos \theta) )</td>
</tr>
<tr>
<td>3, 1</td>
<td>( \sqrt{\frac{15}{2\pi}} (5 \cos^3 \theta - 3 \cos \theta) \sin \phi )</td>
</tr>
<tr>
<td>3, 2</td>
<td>( \sqrt{\frac{21}{2\pi}} \cos \theta \sin^2 \phi \cos \phi )</td>
</tr>
<tr>
<td>3, 3</td>
<td>( \sqrt{\frac{35}{2\pi}} \sin^3 \theta \cos \phi \sin \phi )</td>
</tr>
</tbody>
</table>

\[ \csc \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) - m_\ell^2 \csc^2 \theta \Theta(\theta) = -\ell(\ell + 1)\Theta(\theta) \]
Now, we can discuss where hydrogen’s $e$ might be found.

### Traditional naming scheme

#### Spectroscopic notation

<table>
<thead>
<tr>
<th>Letter</th>
<th>$s$</th>
<th>$p$</th>
<th>$d$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of $\ell$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

$s$: sharp, $p$: principle, $d$: diffuse, $f$: fundamental...

- 3d state: $n=3$ & $l = 2$
- 2p state: $n=2$ & $l = 1$

Electron Prob. Densities in the Hydrogen atom, through $n = 3$

State are labeled using spectroscopic notation: $n$, $l$

2d state is possible? No!!

Because, here, $n=2$, $d =2$; Remember $n > d$

### Spectral Lines

Hydrogen’s energies & spectral lines;

A photon is emitted when the electron jumps downward.

$$E_i = E_f$$

$$E_i - E_f = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \frac{1}{n_i^2} - \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \frac{1}{n_f^2}$$

$$= \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

For a photon, $E = hf = h\nu$. Therefore,

$$\frac{hc}{\lambda} = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

or

$$\frac{1}{\lambda} = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2hc} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

$$= 1.097 \times 10^7 \text{ m}^{-1} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$
Chapter. 8
Spin & Atomic Physics

Outline:
- Evidence of Angular Momentum Quantization
- Identical Particles
- The Exclusion Principle
- Multi-electron Atoms & the Periodic Table
- Characteristic X-Rays

It’s open said that in Q.M. there’re only 3 bound-state problems solvable (w/o numerical approximation tech.)
Most real application: multiple system. so, let’s start an atom with multiple electrons

\[
L_{\text{Ground State}} = \sqrt{l(l+1)}\hbar = 0
\]

\[
L_s = \sqrt{l(l+1)}\hbar = 0
\]

\[
L_p = \sqrt{l(l+1)}\hbar = \sqrt{2}\hbar
\]
Ground State:

The Electron is **NOT** Orbiting around the proton

Classical Physics:

The Electron is Orbiting around the Proton

**Orbiting in Classical Physics**

Potential energy of a dipole $\mu$ in a magnetic field $B$

$$U = -\mu \cdot B$$

$$\mu = eIA = \frac{e}{T} \pi r^2 = \frac{e}{2\pi r^2} \pi r^2 = \frac{e}{2} vr = \frac{e}{2m_e} (m_e vr)$$

$$\mu_L = -\frac{e}{2m_e} L$$

**Magnetic Dipole Moment**

Conventional current is opposite to electron motion

$\mu = IA$

$L = \mathbf{r} \times \mathbf{p}$

Fundamental charge

Period of revolution

A charge with angular m/m has a magnetic dipole moment

**Magnetic force on a system with dipole moment $\mu$**

$$F = -\nabla(-\mu \cdot B) = \nabla(\mu_x B_x + \mu_y B_y + \mu_z B_z)$$

$U$

$F$ can be measured

$F = \text{negative gradient of potential energy}$
The Stern-Gerlach Experiment

An atom with a magnetic dipole moment passing through a non-uniform B-field

Classical Expectation

\[ F = \frac{\partial B_z}{\partial z} \hat{z} \]

Quantum Theory Expectation

Important factor governing the effect of B-field; so, magnetic quantum #

Ground State -> \( l = 0 \) -> \( L = 0 \) -> \( F = 0 \)
**Ground State** -> \( l = 0 \) -> \( L = 0 \) -> \( F = 0 \) (???)

**Surprise:** Real Experimental Result

**The Solution:**

INTRINSIC MAGNETIC MOMENT and ANGULAR MOMENTUM called “SPIN”

is “carried” by every electron

\[
\mu_S = -g_e \frac{e}{2m_e} S \quad (g_e \approx 2)
\]

Like for \( L \):

\[
L = \sqrt{\ell (\ell + 1) \hbar}
\]

Intrinsic angular momentum:

\[
S = \sqrt{s(s + 1) \hbar}
\]

\( s \) – the quantum number of SPIN

Intrinsic property of a particle