Quantum Mechanics in One Dimension, Part II

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No Degeneracy Theorem

Theorem: There is no degeneracy in one-dimensional bound states.

Proof: Suppose, there are two TISE solutions corresponding to the same energy:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi_1(x) = E\psi_1(x) \quad (1)$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi_2(x) = E\psi_2(x) \quad (2)$$

Multiply Eq. 1 by $\psi_2(x)$, multiply Eq. 2 by $\psi_1(x)$, and then subtract Eq. 2 from Eq. 1. Obtain

$$\psi_1(x)\frac{d^2}{dx^2}\psi_2(x) - \psi_2(x)\frac{d^2}{dx^2}\psi_1(x) = 0$$

$$\frac{d}{dx}\left(\psi_1(x)\frac{d}{dx}\psi_2(x) - \psi_2(x)\frac{d}{dx}\psi_1(x)\right) = 0$$

$$\psi_1(x)\frac{d}{dx}\psi_2(x) - \psi_2(x)\frac{d}{dx}\psi_1(x) = c$$

No Degeneracy Theorem (Cont'd)

For bound states, $\psi_1(x)$ and $\psi_2(x)$ vanish as $x \to \infty$, so $c = 0$. Therefore, we must have

$$\psi_1(x)\frac{d}{dx}\psi_2(x) = \psi_2(x)\frac{d}{dx}\psi_1(x)$$
$$\frac{1}{\psi_2(x)}\frac{d}{dx}\psi_2(x) = \frac{1}{\psi_1(x)}\frac{d}{dx}\psi_1(x)$$
$$\ln\psi_2(x) = \ln\psi_1(x) + c'$$
$$\psi_2(x) = A\psi_1(x)$$

However, states described by $\psi_1(x)$ and $A\psi_1(x)$ are physically equivalent. Q.E.D.

Note that this theorem works *for bound states only*. It is manifestly false for the free particle.

Real Eigenfunctions Theorem

Theorem: The eigenfunctions of H corresponding to one-dimensional bound states can always be chosen pure real in the coordinate basis.

Proof: If $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$ then, by conjugation, $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi^*(x) = E\psi^*(x)$. Averaging these two equations, $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\frac{\psi(x)+\psi^*(x)}{2} = E\frac{\psi(x)+\psi^*(x)}{2}$. Note that the wavefunction $\frac{\psi(x)+\psi^*(x)}{2}$ is manifestly real and, according to the "no degeneracy theorem", represents a unique state corresponding to energy E.

Existence of a Bound State Theorem

Theorem: Every "attractive" potential in one dimension has at least one bound state.

Proof: Note that $\langle E \rangle = \langle H \rangle \geq E_0$, where E_0 is the lowest possible eigenvalue of the TISE. Consider $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. Suppose, $\psi(x)$ is real and $\lim_{|x|\to\infty} \psi(x) = 0$. Then

$$\left\langle -\frac{d^2}{dx^2} \right\rangle = -\int_{-\infty}^{\infty} \psi(x) \frac{d^2}{dx^2} \psi(x) dx = -\psi(x)\psi'(x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left(\frac{d\psi(x)}{dx} \right)^2 dx$$
$$= \int_{-\infty}^{\infty} \left(\frac{d\psi(x)}{dx} \right)^2 dx$$

Now, consider $\psi_\kappa(x) = \sqrt{\kappa} e^{-\kappa|x|}$. For this function,

$$\left\langle -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \right\rangle = \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left(\frac{d\psi_\kappa(x)}{dx} \right)^2 dx = \frac{\hbar^2 \kappa^2}{2m} \quad (3)$$

Existence of a Bound State Theorem (Cont'd)

At the same time,

$$\langle V(x)\rangle = \kappa \int_{-\infty}^{\infty} e^{-2\kappa|x|} V(x) dx \tag{4}$$

Combining together Eqs. 3 and 4,

$$\langle H\rangle = \frac{\hbar^2\kappa^2}{2m} + \kappa \int_{-\infty}^{\infty} e^{-2\kappa|x|} V(x) dx$$

For small values of κ,

$$\langle H\rangle \approx \kappa \int_{-\infty}^{\infty} V(x) dx$$

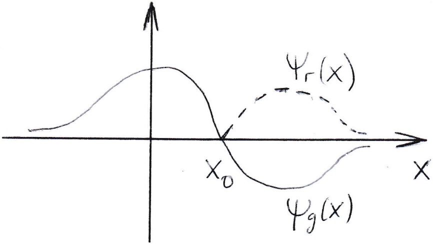
Therefore, $\langle H\rangle$ can always be made negative as long as $\int_{-\infty}^{\infty} V(x)dx < 0$. As $E_0 \leq \langle H\rangle$, $E_0 < 0$. If $\lim_{|x|\to\infty} V(x) = 0$, this is a bound state. Q.E.D.

Note that this theorem effectively *defines* an attractive potential as the potential for which $\int_{-\infty}^{\infty} V(x)dx < 0$ and $\lim_{|x|\to\infty} V(x) = 0$.

No Nodes in the Ground State Theorem

Theorem: Wavefunction that corresponds to the lowest energy bound state (a.k.a. the ground state) of a 1-d system has no nodes in the coordinate representation.

Proof: The wavefunction that corresponds to the ground state, $\psi_g(x)$, must be unique and can be chosen pure real. Suppose, this function has a node. Then we can construct another function, $\psi_r(x)$, by reflecting a part of $\psi_g(x)$ vertically, whenever $\psi_g(x) < 0$ (see figure below). Note that $\psi_r(x) \neq \alpha\psi_g(x)$ for any α, so $\psi_r(x)$ represents a distinct physical state.



No Nodes in the Ground State Theorem (Cont'd)

The following assumes a single node at x_0, but the extension to multiple nodes is obvious. Consider $\langle E_r \rangle = \langle \psi_r | H | \psi_r \rangle$. As ψ_r represents a distinct physical state, we must have $\langle E_r \rangle > E_0 = \langle \psi_g | H | \psi_g \rangle$. On the other hand, in the coordinate basis,

$$\begin{aligned}\langle \psi_g | H | \psi_g \rangle &= \int_{-\infty}^{\infty} \psi_g(x) H \psi_g(x) dx \\ &= \int_{-\infty}^{x_0 - \varepsilon} \psi_g(x) H \psi_g(x) dx + \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \psi_g(x) H \psi_g(x) dx \\ &\quad + \int_{x_0 + \varepsilon}^{\infty} \psi_g(x) H \psi_g(x) dx,\end{aligned}$$

with a similar decomposition for $\langle \psi_r | H | \psi_r \rangle$. The following should be obvious:

$$\int_{-\infty}^{x_0 - \varepsilon} \psi_g(x) H \psi_g(x) dx = \int_{-\infty}^{x_0 - \varepsilon} \psi_r(x) H \psi_r(x)$$

$$\int_{x_0 + \varepsilon}^{\infty} \psi_g(x) H \psi_g(x) dx = \int_{x_0 + \varepsilon}^{\infty} \psi_r(x) H \psi_r(x) dx$$

$$\lim_{\varepsilon \to 0} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \psi_g(x) H \psi_g(x) dx = 0$$

No Nodes in the Ground State Theorem (Cont'd)

But what is $\lim_{\varepsilon \to 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_r(x) H \psi_r(x) dx$? As $\psi_r(x)$ has a discontinuous first derivative at x_0 and an infinite second derivative, this is not immediately apparent. Consider, however

$$-\int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_r(x) \frac{d^2}{dx^2} \psi_r(x) dx = -\psi_r(x)\psi_r'(x)\Big|_{x_0-\varepsilon}^{x_0+\varepsilon} + \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left(\frac{d\psi_r(x)}{dx}\right)^2 dx$$

As $\left(\frac{d\psi_r(x)}{dx}\right)^2 = \left(\frac{d\psi_g(x)}{dx}\right)^2$, $\lim_{\varepsilon \to 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left(\frac{d\psi_r(x)}{dx}\right)^2 dx = 0$. On the other hand,

$$\begin{aligned}\lim_{\varepsilon \to 0} \left(-\psi_r(x)\psi_r'(x)\Big|_{x_0-\varepsilon}^{x_0+\varepsilon}\right) &= -\psi_r(x_0)(\psi_r'(x_0+0) - \psi_r'(x_0-0)) \\ &= 2\psi_g(x_0)\psi_g'(x_0) = 0\end{aligned}$$

because x_0 is a node. Therefore, $\lim_{\varepsilon \to 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_r(x) H \psi_r(x) dx = 0$, and we have $\langle E_r \rangle = E_0$. This is impossible, so the assumption about existence of the node must be incorrect. Q.E.D.

No Nodes in the Ground State Theorem (Cont'd)

Corollary: The bound state eigenfunctions corresponding to TISE eigenvalues other than E_0 must have at least one node.

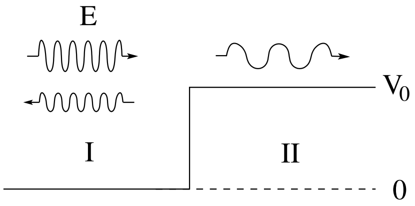
Proof: This is obvious from the orthogonality condition $\int_{-\infty}^{\infty} \psi_n(x)\psi_g(x)dx = 0$. This condition can not be satisfied if both $\psi_n(x)$ and $\psi_g(x)$ have no nodes. In this case the integrand has the same sign everywhere, and $\left|\int_{-\infty}^{\infty} \psi_n(x)\psi_g(x)dx\right| > 0$.

Note that, for 1-d bound states, the wavefunction corresponding to the n^{th} energy level (counting from $n = 0$ for the ground state) has exactly n nodes. This powerful statement is known as the *oscillation theorem* or the *nodal theorem*. Proof of this theorem is beyond the scope of this course.

Special Properties of 1-d Bound States (a Summary)

Please keep in mind the four theorems about 1-d bound states discussed above:

- No degeneracy
- Real eigenfunctions
- Existence of a bound state for an attractive potential
- No nodes in the ground state



- The single-step potential is

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & x > 0 \end{cases}$$

- Shankar performs a detailed treatment of a Gaussian packet scattering off the single-step potential. We will limit the discussion to a monochromatic wave. This example is already sufficient to illustrate all basic concepts.
- Our main goal will be to determine the *transmission* and *reflection coefficients*. These coefficients are defined as the ratios of the probability fluxes, e.g., the transmission coefficient T is $j_{\rm II}/j_{\rm I,in}$.

Wavefunction for the Single-Step Potential

- The TISE for this problem is

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}(V(x) - E)\psi(x)$$

Assuming that $E > V_0$, the general solution of this equation is

$$\psi(x) = \begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x}, & x < 0 \ \ \text{(region I)} \\ Ce^{ik_2 x} + De^{-ik_2 x}, & x > 0 \ \ \text{(region II)} \end{cases},$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. The coefficients k_1 and k_2 are sometimes called wave numbers.

- We can simplify the expression for $\psi(x)$ using physics arguments:

$$\psi(x) = \begin{cases} e^{ik_1 x} + Be^{-ik_1 x}, & x < 0 \ \ \text{(region I)} \\ Ce^{ik_2 x}, & x > 0 \ \ \text{(region II)} \end{cases} \tag{5}$$

Solving the TISE for the Single-Step Potential

- The coefficients B and C are determined using the wavefunction continuity conditions at $x = 0$: $\psi(-0) = \psi(+0)$ and $\psi'(-0) = \psi'(+0)$.

$$1 + B = C \qquad \text{from the continuity of the wavefunction}$$
$$ik_1(1 - B) = ik_2 C \qquad \text{from the continuity of the derivative}$$

- This obviously results in $ik_1(1 - B) = ik_2(1 + B)$ and then

$$B = \frac{k_1 - k_2}{k_1 + k_2} = \frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}} \tag{6}$$

$$C = \frac{2k_1}{k_1 + k_2} = \frac{2\sqrt{E}}{\sqrt{E} + \sqrt{E - V_0}} \tag{7}$$

Transmission and Reflection Coefficients

Reminder: the probability flux for $\psi(x) = Ae^{ikx}$ is $j = |A|^2 \frac{k\hbar}{m} = |A|^2 v$.

$$T = \frac{j_{\text{II}}}{j_{\text{I,in}}} = \frac{|C|^2 v_{\text{II}}}{v_{\text{I}}} = \frac{|C|^2 \sqrt{E - V_0}}{\sqrt{E}} = \frac{4\sqrt{E}\sqrt{E - V_0}}{\left|\sqrt{E} + \sqrt{E - V_0}\right|^2} \tag{8}$$

$$R = \frac{j_{\text{I,out}}}{j_{\text{I,in}}} = |B|^2 = \left|\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right|^2 \tag{9}$$

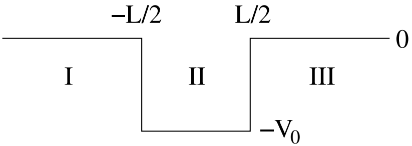
- The probability flux is conserved, i.e., $j_{\text{II}} + j_{\text{I,out}} = j_{\text{I,in}}$. This means that we must have $T + R = 1$. Check that this is indeed the case!
- $\lim_{E \to \infty} T = \lim_{V_0 \to 0} T = 1$.
- $\lim_{E \to V_0} T = 0$.
- What happens if $V_0 < 0$? What is $\lim_{V_0 \to -\infty} T$?

Penetration Depth

- Consider the single-step potential problem with $E < V_0$. Do we have to solve it differently?
- It turns out that the solution given by Eqs. 5, 6, and 7 works just fine if we allow for imaginary $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$ (note that we need to choose the correct square root). The wavefunction in region II is now $Ce^{-|k_2|x}$. The probability density is then $p(x) \propto |C|^2 e^{-2|k_2|x}$.
- The quantity $d = \frac{1}{2|k_2|}$ is called penetration depth. In terms of the penetration depth, $p(x) \propto e^{-x/d}$.
- The reflection coefficient from Eq. 9 becomes 1.
- The sum of the transmission coefficient from Eq. 8 and the reflection coefficient from Eq. 9 is no longer 1. What is going on? Is the probability flux still conserved?

Going (Slightly) Beyond Shankar's Ch. 5

Finite Depth Square Well



- The finite depth square well potential is

$$V(x) = \begin{cases} -V_0 & \text{if } |x| < L/2 \\ 0 & \text{if } |x| > L/2 \end{cases}$$

- We will be interested in the bound states of this system, $E < 0$ case.
- Due to the parity argument, the TISE solutions must be even or odd:

$$\psi_{\text{even}}(x) = \begin{cases} Ce^{-\kappa|x|}, & |x| > \frac{L}{2} \quad \text{(regions I and III)} \\ A\cos kx, & |x| < \frac{L}{2} \quad \text{(region II)} \end{cases}$$

$$\psi_{\text{odd}}(x) = \begin{cases} -Ce^{\kappa x}, & x < -\frac{L}{2} \quad \text{(region I)} \\ B\sin kx, & |x| < \frac{L}{2} \quad \text{(region II)} \\ Ce^{-\kappa x}, & x > \frac{L}{2} \quad \text{(region III)} \end{cases}$$

Solving the TISE for the Finite Depth Square Well

- To determine $\psi(x)$, we need to impose the wavefunction continuity conditions at $x = \frac{L}{2}$: $\psi(\frac{L}{2} - 0) = \psi(\frac{L}{2} + 0)$ and $\psi'(\frac{L}{2} - 0) = \psi'(\frac{L}{2} + 0)$. The continuity conditions at $x = -\frac{L}{2}$ will be satisfied automatically due to the wavefunction symmetry.
- For the even wavefunction the continuity conditions give

$$A \cos \frac{kL}{2} = Ce^{-\frac{\kappa L}{2}} \qquad \text{from the continuity of the wavefunction} \tag{10}$$

$$-kA \sin \frac{kL}{2} = -\kappa Ce^{-\frac{\kappa L}{2}} \quad \text{from the continuity of the derivative} \tag{11}$$

Now, divide Eq. 11 by Eq. 10 and obtain

$$k \tan \frac{kL}{2} = \kappa \tag{12}$$

For the odd wavefunction similar reasoning generates the condition

$$k \cot \frac{kL}{2} = -\kappa \tag{13}$$

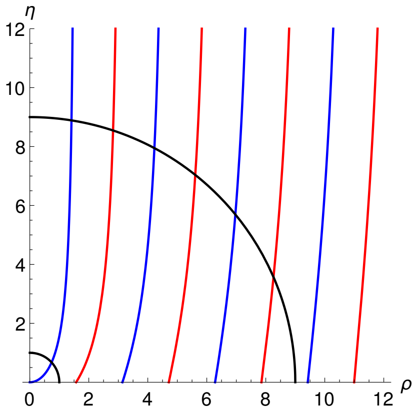
Solving the TISE for the Finite Depth Square Well (Cont'd)

- While Eqs. 12 and 13 can be used to define κ in terms of k, they are by themselves insufficient to pinpoint possible values of k and κ. Note, however, that $\kappa^2 = -\frac{2mE}{\hbar^2}$ while $k^2 = \frac{2m(E+V_0)}{\hbar^2}$. This means that $\kappa^2 + k^2 = \frac{2mV_0}{\hbar^2}$. In combination with Eq. 12 or 13, this constraint gives us the ability to find k and κ.
- It is more convenient to work with dimensionless variables $\rho = \frac{kL}{2}$ and $\eta = \frac{\kappa L}{2}$. In terms of these variables, we have

$$\begin{aligned} \eta &= \rho \tan \rho && \text{for even eigenfunctions} \\ \eta &= -\rho \cot \rho && \text{for odd eigenfunctions} \\ \rho^2 + \eta^2 &= \frac{mV_0 L^2}{2\hbar^2} && \text{for the constraint} \end{aligned}$$

- While these equations can not be solved in terms of simple algebraic functions, a pretty good idea about possible solutions can be obtained by plotting η curves defined by these formulae as a function of ρ.

Graphical TISE Solution for the Finite Depth Square Well



$\eta = \rho \tan \rho$	even eigenfunctions
$\eta = -\rho \cot \rho$	odd eigenfunctions
$\rho^2 + \eta^2 = \frac{mV_0 L^2}{2\hbar^2}$	constraint

The constraint equation is shown for couple different values of V_0 (black line)

Are possible values of k consistent with the infinite square well as $V_0 \to \infty$?

What is the smallest possible value of V_0 that allows for two bound states in the square well potential? What is the energy of the second bound state in this case?

Example Inverse Problem

August 2020 Prelim problem D1P4

A particle of mass m is moving in one dimension in the potential $V(x)$. The particle is in an eigenstate of the Hamiltonian, with probability density for the position given by $\rho(x) = \frac{2a^3}{\pi\left(x^2+a^2\right)^2}$, where a is a positive parameter.

(a) (30 points) Determine the wave function $\psi(x)$ from $\rho(x)$. Argue that the solution is unique (up to an overall phase factor).

(b) (20 points) Is the particle in the ground state? Explain your reasoning.

(c) (50 points) Determine $V(x)$.

Solution of the Example Inverse Problem

- Part (a). Assuming $\psi(x)$ continuity,

$$\psi(x) = \sqrt{\rho(x)} = \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2 + a^2}$$

  Why this solution is unique?
- Part (b). Is this the ground state?
- Part (c). Use the 1-d TISE:

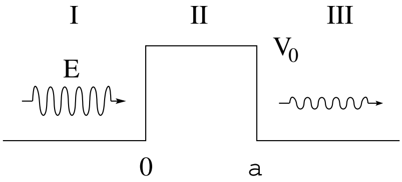
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E_0\psi(x)$$

  From this equation (check!),

$$V(x) = E_0 + \frac{1}{\psi(x)}\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E_0 + \frac{\hbar^2}{m}\frac{3x^2 - a^2}{(x^2 + a^2)^2}$$

  As E_0 is arbitrary, this potential is defined up to an additive constant.
- Is this an attractive potential? Note that $\int_{-\infty}^{\infty} \frac{3x^2-1}{(x^2+1)^2} dx = \pi$. Is there a contradiction with the "existence of a bound state" theorem?

Tunneling Through a Potential Barrier



- Consider the rectangular potential:

$$V(x) = \begin{cases} 0, & x < 0 \text{ or } x > a \\ V_0, & 0 < x < a \end{cases}$$

  We will be interested in the transmission coefficient.

- What are some possible physical realizations of this potential?
- If we shift the potential to the interval $[-a/2, a/2]$, can parity be useful?

Solving the TISE for the Rectangular Barrier

- Unlike the case of finite depth square well, here we are interested in the solutions with $E > 0$.
- Even though $[\Pi, H] = 0$ for the shifted potential, parity is not useful in this problem because the conditions at $x \to -\infty$ and $x \to \infty$ are physically different. Both incoming and reflected waves propagate at $x \to -\infty$, while at $x \to \infty$ only the transmitted wave exists.
- We will search for the TISE solution in the form

$$\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx}, & x < 0 & \text{(region I)} \\ Be^{i\kappa x} + Ce^{-i\kappa x}, & 0 < x < a & \text{(region II)} \\ De^{ik(x-a)}, & x > a & \text{(region III)} \end{cases}$$

  where $k = \frac{\sqrt{2mE}}{\hbar}$ and $\kappa = \frac{\sqrt{2m(E-V_0)}}{\hbar}$, allowing for pure imaginary κ.
- The transmission coefficient $T = |D|^2$ (velocities are the same in region I and region III).

Solving the TISE for the Rectangular Barrier (Cont'd)

- The wavefunction continuity conditions at $x = 0$:

$$1 + A = B + C \tag{14}$$

$$k(1 - A) = \kappa(B - C) \tag{15}$$

The wavefunction continuity conditions at $x = a$:

$$Be^{i\kappa a} + Ce^{-i\kappa a} = D \tag{16}$$

$$\kappa(Be^{i\kappa a} - Ce^{-i\kappa a}) = kD \tag{17}$$

- Eliminate D by multiplying Eq. 16 with k and subtracting Eq. 17. This gives

$$k(Be^{i\kappa a} + Ce^{-i\kappa a}) = \kappa(Be^{i\kappa a} - Ce^{-i\kappa a}) \tag{18}$$

- Solve Eq. 18 for C in terms of B. Obtain $C = \frac{\kappa - k}{\kappa + k} e^{2i\kappa a} B$. We can now say that

$$C = \alpha B, \tag{19}$$

where $\alpha = \frac{\kappa - k}{\kappa + k} e^{2i\kappa a}$.

Solving the TISE for the Rectangular Barrier (Cont'd)

- Substitute Eq. 19 into Eqs. 14 and 15. Obtain

$$1 + A = (1 + \alpha)B \tag{20}$$

$$k(1 - A) = \kappa(1 - \alpha)B \tag{21}$$

- Divide Eq. 21 by Eq. 20. Obtain

$$k\frac{1 - A}{1 + A} = \kappa\frac{1 - \alpha}{1 + \alpha} \tag{22}$$

- Solve Eq. 22 for A. Obtain

$$A = \frac{1 - \frac{\kappa}{k}\frac{1-\alpha}{1+\alpha}}{1 + \frac{\kappa}{k}\frac{1-\alpha}{1+\alpha}} \tag{23}$$

Solving the TISE for the Rectangular Barrier (Cont'd)

- It is convenient to introduce $\beta = \frac{\kappa}{k}$. Note that $\alpha = \frac{\beta-1}{\beta+1} e^{2i\kappa a}$ and that

$$\frac{1-\alpha}{1+\alpha} = \frac{\beta + 1 - (\beta - 1)e^{2i\kappa a}}{\beta + 1 + (\beta - 1)e^{2i\kappa a}} = \frac{(\beta + 1)e^{-i\kappa a} - (\beta - 1)e^{i\kappa a}}{(\beta + 1)e^{-i\kappa a} + (\beta - 1)e^{i\kappa a}} = \frac{\cos \kappa a - i\beta \sin \kappa a}{\beta \cos \kappa a - i \sin \kappa a} \quad (24)$$

- Substitute Eq. 24 into Eq. 23. Obtain

$$A = \frac{(1 - \beta^2) \sin \kappa a}{(1 + \beta^2) \sin \kappa a + 2i\beta \cos \kappa a} = \frac{(k^2 - \kappa^2) \sin \kappa a}{(k^2 + \kappa^2) \sin \kappa a + 2ik\kappa \cos \kappa a}$$

- We can now calculate $B = \frac{1+A}{1+\alpha}$, $C = \alpha B$, etc, and solve for the complete $\psi(x)$. However, knowledge of A is already sufficient for determination of transmission and reflection coefficients.

Transmission Coefficient for the Rectangular Barrier

- Note that $\cos ix = \operatorname{ch} x$, $\sin ix = i \operatorname{sh} x$, $ch^2 x - sh^2 x = 1$.
- The reflection coefficient is

$$R = |A|^2 = \begin{cases} \frac{(k^2-\kappa^2)^2 \sin^2 \kappa a}{(k^2+\kappa^2)^2 \sin^2 \kappa a + 4k^2\kappa^2 \cos^2 \kappa a}, & E > V_0, \text{ real } \kappa \\ \frac{(k^2+|\kappa|^2)^2 \operatorname{sh}^2 |\kappa| a}{(k^2-|\kappa|^2)^2 \operatorname{sh}^2 |\kappa| a + 4k^2|\kappa|^2 \operatorname{ch}^2 |\kappa| a}, & E < V_0, \text{ imaginary } \kappa \end{cases}$$

- The transmission coefficient is

$$T = 1 - R = \begin{cases} \frac{4k^2\kappa^2}{(k^2+\kappa^2)^2 \sin^2 \kappa a + 4k^2\kappa^2 \cos^2 \kappa a}, & E > V_0 \\ \frac{4k^2|\kappa|^2}{(k^2-|\kappa|^2)^2 \operatorname{sh}^2 |\kappa| a + 4k^2|\kappa|^2 \operatorname{ch}^2 |\kappa| a}, & E < V_0 \end{cases}$$

Finally, in terms of E and V_0,

$$T = \begin{cases} \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2 \sin^2\left(\frac{\sqrt{2m(E-V_0)}}{\hbar} a\right)}, & E > V_0 \\ \frac{4E(V_0-E)}{4E(V_0-E) + V_0^2 \operatorname{sh}^2\left(\frac{\sqrt{2m(V_0-E)}}{\hbar} a\right)}, & E < V_0 \end{cases} \qquad (25)$$

Properties of the Transmission Coefficient

- $\lim_{E\to\infty} T = \lim_{V_0\to 0} T = 1.$
- Note that $T = 1$ also for the case $E > V_0$, $\kappa a = n\pi$.
- $\lim_{E\to V_0} T = \frac{1}{1+\frac{ma^2 V_0}{2\hbar^2}}$.
- Consider tunneling through a wide barrier, $E < V_0$ and $\frac{\sqrt{2m(V_0-E)}}{\hbar} a \gg 1$. For $x \gg 1$, $\operatorname{sh} x \approx \frac{1}{2}e^x$ and $\operatorname{sh}^2 x \approx \frac{1}{4}e^{2x}$. Then

$$T \approx \frac{16E(V_0 - E)}{V_0^2} \exp\left(-2\frac{\sqrt{2m(V_0 - E)}}{\hbar} a\right) \tag{26}$$

If, in addition, $E \ll V_0$ then

$$T \approx \frac{16E}{V_0} \exp\left(-2\frac{\sqrt{2mV_0}}{\hbar} a\right) \tag{27}$$

Exponential suppression of the transmission coefficient with barrier width is typical for many barrier types.