Now, for $(\vec{p}_2\cdot\nabla)\hat{r}$ term, spherical coordinates make it easy. $\hat{r} = (1)\hat{r}$ = constant, so all derivatives vanish

(see link I sent you for $(\vec{A}\cdot\nabla)\vec{B}$ in spherical)

in fact, all you get is $\frac{p_{2\theta}}{r}\hat{\theta} + \frac{p_{2\varphi}}{r}\hat{\varphi}$

add & subtract $\frac{p_{2r}\hat{r}}{r}$, get

$$\frac{\vec{p}_2}{r} - p_{2r}\frac{\hat{r}}{r} = \frac{\vec{p}_2}{r} - \frac{(\vec{p}_2\cdot\hat{r})\hat{r}}{r}$$

so

$$\left(\vec{p}_1\cdot\frac{\hat{r}}{r^3}\right)\left(\frac{\vec{p}_2}{r} - \frac{(\vec{p}_2\cdot\hat{r})\hat{r}}{r}\right) = \frac{(\vec{p}_1\cdot\hat{r})\vec{p}_2}{r^4} - \frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{r^4}\hat{r}$$

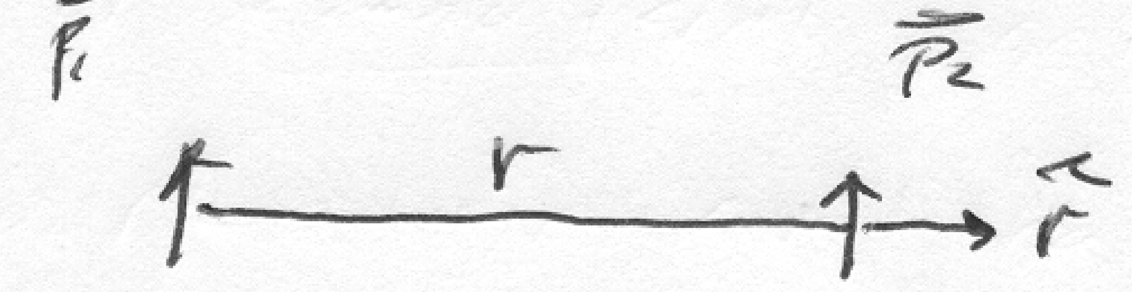
So

$$\vec{F}_2 = \frac{3}{4\pi\varepsilon_0 r^4}\Big[(\vec{p}_2\cdot\hat{r})\vec{p}_1 - 4(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\hat{r} + (\vec{p}_1\cdot\hat{r})\vec{p}_2 - (\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\hat{r} + \vec{p}_1\cdot\vec{p}_2\,\hat{r}\Big]$$

$$\vec{F}_2 = \frac{3}{4\pi\varepsilon_0 r^4}\Big[\big(\vec{p}_1\cdot\vec{p}_2 - 5(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\big)\hat{r} + \vec{p}_1(\vec{p}_2\cdot\hat{r}) + \vec{p}_2(\vec{p}_1\cdot\hat{r})\Big]$$

cases

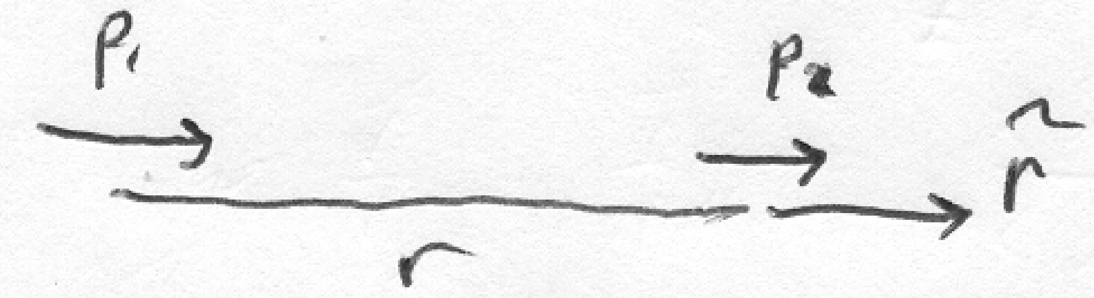
a)



$\vec{p}_1\cdot\vec{p}_2 = p_1p_2$ $\quad \vec{p}_1\cdot\hat{r} = \vec{p}_2\cdot\hat{r} = 0$

So $\vec{F}_2 = \frac{3p_1p_2}{4\varepsilon_0\pi r^4}\hat{r}$

b)



$\vec{p}_1\cdot\vec{p}_2 = p_1p_2$, $\vec{p}_1\cdot\hat{r} = p_1$ $\quad \vec{p}_2\cdot\hat{r} = p_2$

$\vec{p}_1 = p_1\hat{r}$ $\quad \vec{p}_2 = p_2\hat{r}$

$$\vec{F}_2 = \frac{3}{4\pi\varepsilon_0 r^4}\Big[-5p_1p_2\hat{r} + p_1p_2\hat{r} + p_1p_2\hat{r} + p_1p_2\hat{r}\Big]$$

$$\vec{F}_2 = -\frac{3p_1p_2}{2\pi\varepsilon_0 r^4}$$