5. $\vec{E} = (\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}) e^{-(\alpha x + \beta y + \gamma z)}$

a) must have $\nabla \times \vec{E} = 0$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha e^{-()} & \beta e^{-()} & \gamma e^{-()} \end{vmatrix} = \hat{x}\left[\cancel{-\gamma\beta e^{-()} - (-\beta\gamma) e^{-()}}^{\,0}\right] + \hat{y}(0) + \hat{z}(0) = \underline{0}$$

$(() = \alpha x + \beta y + \gamma z)$

similar cancellation

So it is a possible $\vec{E}$ field.

b) Need $\Delta\phi = -\int_i^f \vec{E}\cdot d\vec{s}$, a path integral.

Easiest to re-express $\vec{E}$; $\vec{C} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$;

then
$$\vec{E} = \vec{C} e^{-\vec{C}\cdot\vec{r}} = \vec{C} e^{-(\vec{C}\cdot\hat{r})r}$$

and take $d\vec{s} = dr\,\hat{r}$

Then $\Delta\phi = -\int_0^r \vec{C} e^{-(\vec{C}\cdot\hat{r})r} \cdot dr\,\hat{r}$

$$= -\int_0^r (\vec{C}\cdot\hat{r}) e^{-(\vec{C}\cdot\hat{r})r}\, dr$$

$$\Delta\phi = -\frac{\vec{C}\cdot\hat{r}}{-\vec{C}\cdot\hat{r}} e^{-(\vec{C}\cdot\hat{r})r}\Big|_0^r + \text{const.}$$

so $\Delta\phi = \underline{\phi = e^{-\vec{C}\cdot\vec{r}} + \text{constant}}$

c) $\nabla^2\phi = \nabla\cdot\nabla\phi$ with $\phi = e^{-(\alpha x + \beta y + \gamma z)} + \text{const.}$

$$\nabla\phi = (-\alpha\hat{x} - \beta\hat{y} - \gamma\hat{z}) e^{-(\alpha x + \beta y + \gamma z)}$$

$$\nabla\cdot\nabla\phi = \nabla^2\phi = (\alpha^2 + \beta^2 + \gamma^2) e^{-(\alpha x + \beta y + \gamma z)}$$

so $\nabla^2\phi = 0$ if $\underline{\alpha^2 + \beta^2 + \gamma^2 = 0}$

One or more must be imaginary!