

The Damped Driven Pendulum and Applications

Term Paper By
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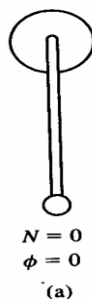
The physics of the damped driven pendulum is based on the dynamics of the simple pendulum. Hence it is very important to understand the dynamics of the simple pendulum. For a simple pendulum of length R and mass m , the angular acceleration of the pendulum is produced by the restoring gravitational torque $mgR \sin \phi$.

Corresponding to the equation of motion

$$mR^2 \frac{d^2 \phi}{dt^2} + mgR \sin \phi = 0$$

Where $I = mR^2$

Now we consider the motion of the pendulum in the following cases as shown where the pendulum is subjected to an applied torque N and is permitted to rotate through its full range of motion. In addition, the pendulum is subject to damping by the viscosity η of the medium in which it rotates. Here we assume the rod as massless and all the pendulum mass is concentrated at the end of the rod. If N is zero then ϕ is zero for the system below.

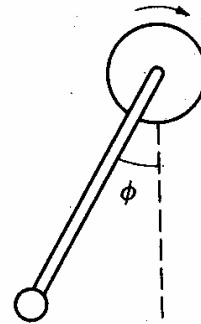


For small angular displacements, the approximation $\sin \phi \approx \phi$ linearises the problem by

making the torque proportional to the displacement and the motion is simple harmonic $\phi = \phi_0 \sin \omega t$ with the characteristic frequency

$$\omega_0 = \left(\sqrt{\frac{g}{R}} \right)$$

For the system below, we can say the following,



$$N = \frac{1}{2} mgR$$

$$\phi = 30^\circ$$

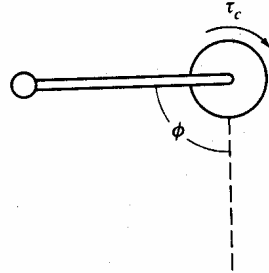
(b)

If a torque N is applied to a stationary pendulum, it will swing out through an angle ϕ . The force of gravity acting on the mass m provides the restoring torque $mgR \sin \phi$ and the pendulum assumes an equilibrium position at the angle ϕ given by

$$N = mgR \sin \phi \quad \left(\frac{d\phi}{dt} = 0 \right)$$

The greater the torque, larger the angle ϕ .

There is a critical torque N_c for the system shown below for which the angle ϕ assumes the value $\frac{\pi}{2}$



$$N = mgR = N_c$$

$$\phi = 90^\circ$$

(c)

The Critical torque is

$$N_c = mgR$$

If the torque exceeds this critical value, then the applied torque becomes larger than the restoring torque, $N > mgR \sin \phi$ for all angles ϕ . As a result, the pendulum will begin to rotate beyond $\phi = \frac{\pi}{2}$ and it will continue to rotate as long as the torque $N > N_c$ is applied. The motion will take place at a variable angular speed ω and it can persist if the torque is later removed.

$$\omega = \frac{d\phi}{dt}$$

Now we examine the case of the damped pendulum assuming that the damping force $F_{damp} = \omega$ is proportional to the angular velocity ω . The differential equation is as below,

with the restoring force $mgR \sin \phi$ and the damping torque $\eta \frac{d\phi}{dt}$ added.

$$N = mR^2 \frac{d^2\phi}{dt^2} + \eta \frac{d\phi}{dt} + mgR \sin \phi$$

If we define a critical frequency ω_c corresponding to the angular speed at which the damping torque $\eta \omega$ equals the critical torque mgR

$$\omega_c = \frac{mgR}{\eta} = \frac{N_c}{\eta}$$

Then the pendulum equation in the normalized form is as

$$\frac{N}{N_c} = \frac{1}{\omega_0^2} \frac{d^2\phi}{dt^2} + \frac{1}{\omega_c} \frac{d\phi}{dt} + \sin \phi$$

The solutions of the above equation exhibit complex time variations of the angular position $\phi(t)$.

When a constant torque is applied to the pendulum at rest, there will be a initial transient behavior that eventually settles down to a dynamic steady state after the transient die out. Several cases of this dynamic steady state are examined as follows.

1. For low applied torque, $N \leq N_c$ there is a steady state

$$N = N_c \sin \phi$$

in which all the time derivatives vanish after the initial oscillations have dies out. This is shown in

the fig(ii) with the pendulum stationary at the angle .

2. For undamped motion ($\eta = 0$) with a constant applied torque, N, we have the following

$$torque = N - mgR \sin \phi = mR^2 \frac{d^2 \phi}{dt^2}$$

We see that the acting torque is angularly dependent. This torque has special values at the four particular angles.

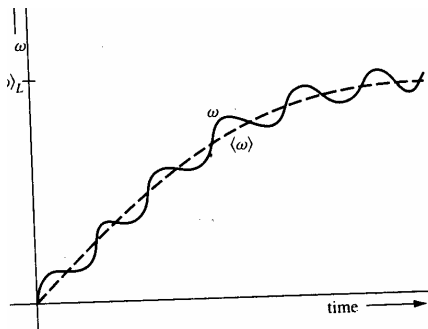
$$torque = N \quad \phi = 0$$

$$torque = N - N_c \quad \phi = \frac{\pi}{2}$$

$$torque = N \quad \phi = \pi$$

$$torque = N + N_c \quad \phi = 3\frac{\pi}{2}$$

If the applied torque N exceeds the critical torque N_c , the motion will be continuously accelerated rotation, and the pendulum increases its energy as times goes on. The angular speed also increases with time, but with fluctuations that repeat every cycle, as indicated in figure ()



The average over these oscillations provides the average angular speed

$$\langle \omega \rangle = \left\langle \frac{d\phi}{dt} \right\rangle$$

this continually increases linearly with the time.

3. When damping is present with $\omega_c \ll \omega_0$ and $N > N_c$ the angular speed continues to increase until the damping term $\eta \frac{d\phi}{dt}$ approaches the value of the applied torque.

When this occurs, the average angular speed $\langle \omega \rangle$ approaches a limiting value $\langle \omega \rangle_L$ as shown in the fig () and the acceleration fluctuates around an average that is zero.

$$\left\langle \frac{d\phi^2}{dt^2} \right\rangle = 0$$

The pendulum undergoes what is called quasi static motion, rotating with the angular speed that undergoes periodic variations but always remains close to the average $\langle \omega \rangle_L$. Neglecting the acceleration term in the equation of motion, we have

$$\frac{N}{N_c} = \frac{1}{\omega_c} \frac{d\phi}{dt} + \sin \phi$$

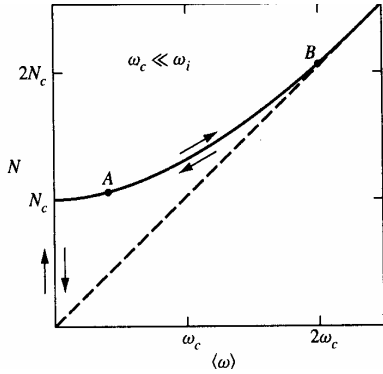
this is an equation that can be solved analytically with the solutions

$$\langle \omega \rangle = 0 \text{ for } N < N_c$$

$$\langle \omega \rangle = \omega_c \sqrt{\left[\left(\frac{N}{N_c} \right)^2 - 1 \right]} \text{ for } N > N_c$$

$$\langle \omega \rangle = \left(\frac{N}{N_c} \right) \omega_c \text{ for } N \gg N_c$$

which are shown in the fig ()

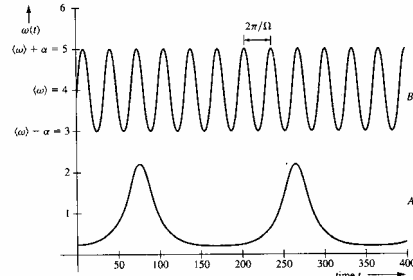


The actual cyclic variations in ω for points A and B on this plot are shown in the fig (). At point A, the applied torque has the value $N = 1.2N_c$, so from equations And the net torque varies between $0.2N_c$ and $2.2N_c$ around the cycle and the angular speed is fast at the bottom and slow at the top, with the variations shown at the lower part of the fig()

For point B, we have $N = 2N_c$ so the net torque varies between N_c and $3N_c$, producing the more regular variations in angular speed presented at the top of the fig (). In the limit $N \gg N_c$, meaning $\langle \omega \rangle \gg \omega_c$, the angular speed begins to approximate a sinusoidal variation with time

$$\omega(t) \approx \langle \omega \rangle + \alpha \sin \Omega t$$

Which approximates point B in fig ().



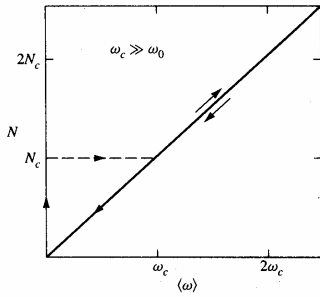
4. For the negligible damping case ($\eta \rightarrow 0$ and $\omega_c \gg \omega_0$), the steady state solution can still occur for $N < N_c$ with the pendulum held fixed at the angle ϕ defined which means that $\omega = \langle \omega \rangle = 0$

In addition, the solution, in which the torque balances the time averaged, damping force, now applies for all values of N, both less than and greater than N_c and so we have

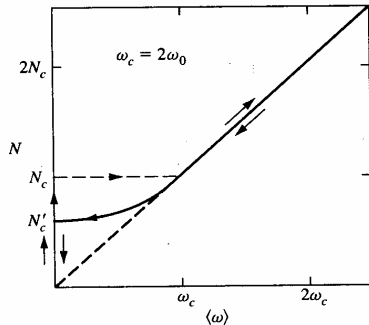
$$\omega = 0 \text{ for } N \leq N_c$$

$$\langle \omega \rangle = \left(\frac{N}{N_c} \right) \omega_c \text{ for } 0 \leq N$$

These are plotted in fig(). The system exhibits hysteresis, meaning that the behavior differs for increasing and decreasing torque. When the torque is increased for $N < N_c$, the pendulum stabilizes at the angle ϕ satisfies the relation $N = N_c \sin \phi$ so $\omega = 0$. When N reaches the critical torque N_c the angular speed jumps to the value ω_c and then rises linearly with further increases as shown in the figure. For decreasing torque and $\langle \omega \rangle$ remains proportional to N all the way to the origin as shown.



5. Figure () shows the response for $\omega_c \ll \omega_0$ and fig () presents it for $\omega_c \gg \omega_0$ and the question arises as what is the behavior for an intermediate condition such as $\omega_c \approx \omega_0$ this requires solving the general equation () since no approximations can be made. The N versus $\langle \omega \rangle$ characteristic for the particular case $\omega_c = 2\omega_0$ is plotted in the fig ()



We see from the figure that for increasing torques there is the usual initial rise in then at zero frequency until the critical value N_C is reached, at which point the average angular speed jumps to ω_c , as in the $\omega_c \gg \omega_0$ case of the fig (). For decreasing torques. There is a hysteresis with the zero average frequency reached at a torque N'_C which is less than N_C .

Application:

The damped driven pendulum equation has a particularly important applications in *solid state physics*. When 2 superconductors in close proximity with a thin layer of insulating material between them, the arrangement constitutes a *Josephson junction*, which has the property that electric current I can flow across the junction with zero applied voltage, up to a certain critical value I_C . Current exceeding this value is accompanied by the presence of a voltage, and the plots of current I versus voltage V for the junction exhibit *hysteresis*. The Josephson junction satisfies the same differential equation as the damped oscillator with the current playing the role of the torque, the voltage playing the role of the average angular speed, the capacitance acting like a moment of inertia and the electrical conductance serving as the viscosity. The variable, which is the angle.. for the oscillator, becomes the phase difference ψ across the Josephson junction.