UNIT 11 READING A

Read the following excerpt (from Peter J. Nolan, *Fundamentals of College Physics Second Edition*, Wm. C. Brown Publishers, Chicago, 1993):

13.6 The Equation of Continuity

Up to now, we have studied only fluids at rest. ***Let us now study fluids in motion, the subject matter of hydrodynamics.*** The study of fluids in motion is relatively complicated, but the analysis can be simplified by making a few assumptions. Let us assume that the fluid is incompressible and flows freely without any turbulence or friction between the various parts of the fluid itself and any boundary containing the fluid, such as the walls of a pipe. A fluid in which friction can be neglected is called a ***nonviscous fluid.*** A fluid, flowing steadily without turbulence, is usually referred to as being in ***streamline flow.*** The rather complicated analysis is further simplified by the use of two great conservation principles: the conservation of mass, and the conservation of energy. ***The law of conservation of mass results in a mathematical equation, usually called the equation of continuity. The law of conservation of energy is the basis of Bernoulli's theorem,*** the subject matter of section 13.7.

Let us consider an incompressible fluid flowing in the pipe of figure 13.9. At a particular instant of time the small mass of fluid Δm, shown in the left-hand portion of the pipe will be considered. This mass is given by a slight modification of equation 13.2, as

$$\Delta m = \rho \Delta V \quad (13.33)$$

Because the pipe is cylindrical, the small portion of volume of fluid is given by the product of the cross-sectional area A_1 times the length of the pipe Δx_1 containing the mass Δm, that is,

$$\Delta V = A_1 \Delta x_1 \quad (13.34)$$

The length Δx_1 of the fluid in the pipe is related to the velocity v_1 of the fluid in the left-hand pipe. Because the fluid in Δx_1 moves a distance Δx_1 in time Δt, $\Delta x_1 = v_1 \Delta t$. Thus,

$$\Delta x_1 = v_1 \Delta t \quad (13.35)$$

Substituting equation 13.35 into equation 13.34, we get for the volume of fluid,

$$\Delta V = A_1 v_1 \Delta t$$

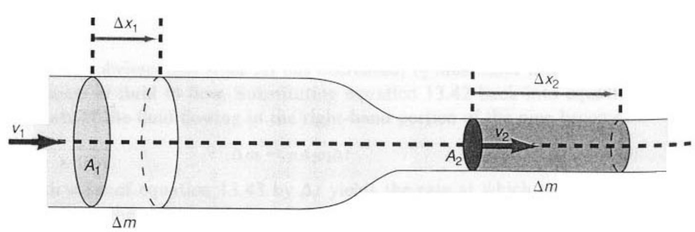

Figure 13.9
The law of conservation of mass and the equation of continuity.

Substituting equation 13.36 into equation 13.33 yields the mass of the fluid as

$$\Delta m = \rho A_1 v_1 \Delta t \tag{13.37}$$

We can also express this as the rate at which the mass is flowing in the left-hand portion of the pipe by dividing both sides of equation 13.37 by Δt, thus

$$\frac{\Delta m}{\Delta t} = \rho A_1 v_1 \tag{13.38}$$

Example 13.15

Flow rate. What is the mass flow rate of water in a pipe whose diameter d is 10.0 cm when the water is moving at a velocity of 0.322 m/s.

Solution

The cross-sectional area of the pipe is

$$A_1 = \frac{\pi d_1^2}{4} = \frac{\pi(0.100 \text{ m})^2}{4}$$
$$= 7.85 \times 10^{-3} \text{ m}^2$$

The flow rate, found from equation 13.38, is

$$\frac{\Delta m}{\Delta t} = \rho A_1 v_1$$
$$= (1.00 \times 10^3 \text{ kg/m}^3)(7.85 \times 10^{-3} \text{ m}^2)(0.322 \text{ m/s})$$
$$= 2.53 \text{ kg/s}$$

Thus 2.53 kg of water flow through the pipe per second.

When this fluid reaches the narrow constricted portion of the pipe to the right in figure 13.9, the same amount of mass Δm is given by

$$\Delta m = \rho \Delta V$$

But since ρ is a constant, the same mass Δm must occupy the same volume ΔV. However, the right-hand pipe is constricted to the narrow cross-sectional area A_2. Thus, the length of the pipe holding this same volume must increase to a larger value Δx_2, as shown in figure 13.9. Hence, the volume of fluid is given by

$$\Delta V = A_2 \Delta x_2 \tag{13.40}$$

The length of pipe Δx_2 occupied by the fluid is related to the velocity of the fluid by

$$\Delta x_2 = v_2 \Delta t$$

Substituting equation 13.41 back into equation 13.40, we get for the volume of fluid,

$$\Delta V = A_2 v_2 \Delta t \tag{13.42}$$

It is immediately obvious that since A_2 has decreased, v_2 must have increased for the same volume of fluid to flow. Substituting equation 13.42 back into equation 13.39, the mass of the fluid flowing in the right-hand portion of the pipe becomes

$$\Delta m = \rho A_2 v_2 \Delta t \tag{13.43}$$

Dividing both sides of equation 13.43 by Δt yields the rate at which the mass of fluid flows through the right-hand side of the pipe, that is,

$$\frac{\Delta m}{\Delta t} = \rho A_2 v_2$$

But ***the law of conservation of mass*** *states that mass is neither created nor destroyed in any ordinary mechanical or chemical process.* Hence, the law of conservation of mass can be written as

Mass flowing into the pipe = mass flowing out of the pipe

or

$$\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta t} \tag{13.45}$$

Thus, setting equation 13.38 equal to equation 13.44 yields

$$\rho A_1 v_1 = \rho A_2 v_2 \tag{13.46}$$

Equation 13.46 is called **the equation of continuity** and is an indirect statement of the law of conservation of mass. Since we have assumed an incompressible fluid, the densities on both sides of equation 13.46 are equal and can be canceled out leaving

$$A_1 v_1 = A_2 v_2 \tag{13.47}$$

Equation 13.47 is a special form of the equation of continuity for incompressible fluids (i.e., liquids).

Applying equation 13.47 to figure 13.9, we see that the velocity of the fluid v_2 in the narrow pipe to the right is given by

$$v_2 = \frac{A_1}{A_2} v_1 \tag{13.48}$$

Because the cross-sectional area A_1 is greater than the cross-sectional area A_2, the ratio A_1/A_2 is greater than one and thus the velocity v_2 must be greater than v_1.

Example 13.16

Applying the equation of continuity. In example 13.15 the cross-sectional area A_1 was 7.85×10^{-3} m² and the velocity v_1 was 0.322 m/s. If the diameter of the pipe to the right in figure 13.9 is 4.00 cm, find the velocity of the fluid in the right-hand pipe.

Solution

The cross-sectional area of the right-hand side of the pipe is

$$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi(0.0400 \text{ m})^2}{4}$$
$$= 1.26 \times 10^{-3} \text{ m}^2$$

The velocity of the fluid on the right-hand side v_2, found from equation 13.48, is

$$v_2 = \frac{A_1}{A_2} v_1 = \left(\frac{7.85 \times 10^{-3} \text{ m}^2}{1.26 \times 10^{-3} \text{ m}^2}\right)(0.322 \text{ m/s})$$
$$= 2.01 \text{ m/s}$$

The fluid velocity increased more than six times when it flowed through the constricted pipe.

Therefore, as a general rule, the equation of continuity for liquids, equation 13.47, says that when the cross-sectional area of a pipe gets smaller, the velocity of the fluid must become greater in order that the same amount of mass *passes a given point in a given time. Conversely, when the cross-sectional area increases, the velocity of the fluid must decrease.* Equation 13.47, the equation of continuity, is sometimes written in the equivalent form

$$Av = \text{constant} \tag{13.49}$$

Example 13.17

Flow rate revisited. What is the flow of mass per unit time for the example 13.16?

Solution

The rate of mass flow for the right-hand side of the pipe, given by equation 13.44, is

$$\begin{aligned}\frac{\Delta m}{\Delta t} &= \rho A_2 v_2 \\ &= (1.0 \times 10^3\ \text{kg/m}^3)(1.26 \times 10^{-3}\ \text{m}^2)(2.01\ \text{m/s}) \\ &= 2.53\ \text{kg/s}\end{aligned}$$

Note that this is the same rate of flow found earlier for the left-hand side of the pipe, as it must be by the law of conservation of mass.

A compressible fluid (i.e., a gas) can have a variable density, and requires an additional equation to specify the flow velocity.