Gauss-Legendre Quadrature

We want to compute the integral $I = \int_a^b f(x)\,dx$ by means of a Gauss-Legendre (GL) quadrature. This assumes that we have the GL quadrature points u_i and weights w_i available for some interval (α to β). Thus, we connect our integral of interest to an equivalent GL integral via

$$\int_a^b f(x)\,dx \longrightarrow \frac{b-a}{\beta-\alpha}\int_\alpha^\beta \phi(u)\,du,$$

where

$$x = \frac{(b-a)u}{(\beta-\alpha)} + \frac{(a\beta - b\alpha)}{(\beta-\alpha)}.$$

Usually, $\alpha = -1$ and $\beta = 1$. If this is the case, then

$$\int_a^b f(x)\,dx \longrightarrow \frac{(b-a)}{2}\int_{-1}^1 f\left[\frac{(b-a)u}{2} + \frac{(b+a)}{2}\right]\,du,$$

or,

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\sum_{i=1}^n f\left[\frac{(b-a)u_i}{2} + \frac{(b+a)}{2}\right]\,w_i.$$

Remember, an n-point GL quadrature is "exact" for $f(x)$ a polynomial of degree $2n-1$.